Lecture 3

The Best-fit Paradigm of FEA: Shadows of the Exact

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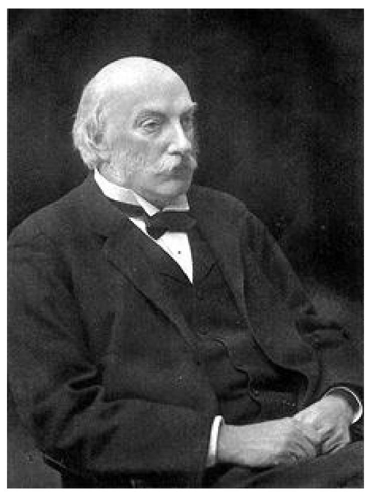
Eminent Scholars of Mathematical and Computational Physics

Carl Friedrich Gauss

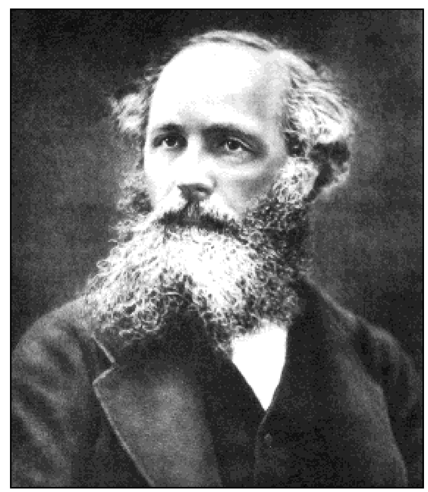
Aldrien Marie Legendre

Joseph Fourier

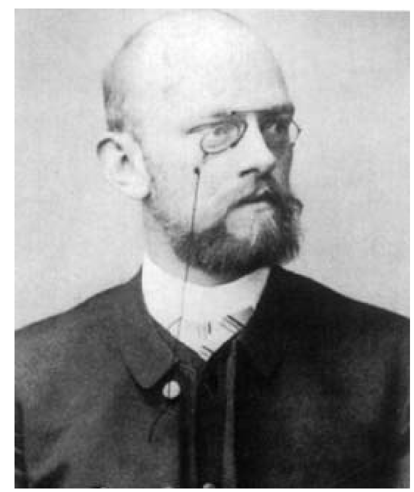
Pierre Simon Laplace



Lord Rayleigh



James Clerk Maxwell



David Hilbert

Lecture 3

The Best-fit Paradigm of FEA: Shadows of the Exact

Chapters

1. The Algebra of Vector Spaces and Projections.

2. How the "Principle of Virtual Work" works to make FEA the best-fit.

3. Real Examples: Simple one-dimensional elements.

4. Consequence of the best-fit nature in FEA.
   Optimal points (Gauss's and Prathap's points) for exact strain recovery.

"The external world of physics has thus become a world of shadows. In removing our illusions we have removed the substance, for indeed we have seen that substance is one of the greatest of our illusions…… Reality is a child which cannot survive without its nurse illusion."

- **Sir Arthur Eddington**

The Nature of the Physical World.

"The whole scientific inquiry starts from the familiar world and in the end it must return to the familiar world; but the part of the journey over which the physicist has charge is in foreign territory."

- **Sir Arthur Eddington**

The Nature of the Physical World.

" The most practical tool is a good theory."

- Albert Einstein

Lecture 3

The Mathematics of Shadows…

Chapter 1

The Algebra of Vector Spaces and Projections

1.1 What are Vectors ?

- ***A vector is a physical quantity that has both magnitude and direction.***

$$\mathbf{A} = \mathbf{i}A_1 + \mathbf{j}A_2 + \mathbf{k}A_3 = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}]\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = [\mathbf{i} \quad \mathbf{j} \quad \mathbf{k}]\{A\}$$

- ***A vector is an ordered array of numbers.***

$$\{A\} = \begin{Bmatrix} A_1 \\ A_2 \\ .. \\ A_n \end{Bmatrix} = [A_1 \quad A_2 \quad .. \quad .. \quad A_n]^T$$

1.2 Linear Independence of Vectors

A set of N vectors are linearly independent if, <u>for not all zero values of</u> α_i

$$\boxed{\sum_{i=1}^{N} \alpha_i \mathbf{A}_i \neq \mathbf{0}} \qquad \textbf{(1.1)}$$

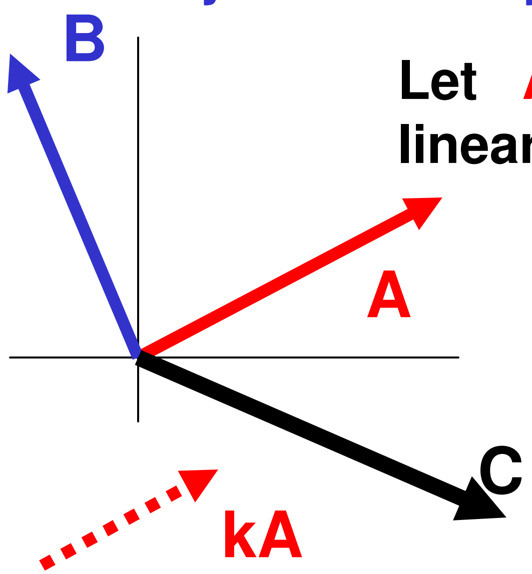
A perfectly flat plane is two-dimensional.

Space, as we perceive, is three dimensional.

What is the logic ?

Why do we say that a flat plane is two-dimensional ?

Only two linearly independent vectors can exist on the 2D plane.



Let A=5i+4j. Note that a=kA=k(5i+4j) and A are not linearly independent because a- kA=0

Let B=-4i+8j. A and B are two linearly independent vectors on the plane, since αA $\neq$ B

Another vector C =13i-12j can be shown as an resultant of a non-trivial linear combination of A and B.

α_1A+ α_2B=C

$$\alpha_1\begin{Bmatrix}5\\4\end{Bmatrix}+\alpha_2\begin{Bmatrix}-4\\8\end{Bmatrix}=\begin{Bmatrix}13\\-12\end{Bmatrix}$$

or

$$\begin{bmatrix}5 & -4\\4 & 8\end{bmatrix}\begin{Bmatrix}\alpha_1\\\alpha_2\end{Bmatrix}=\begin{Bmatrix}13\\-12\end{Bmatrix}$$

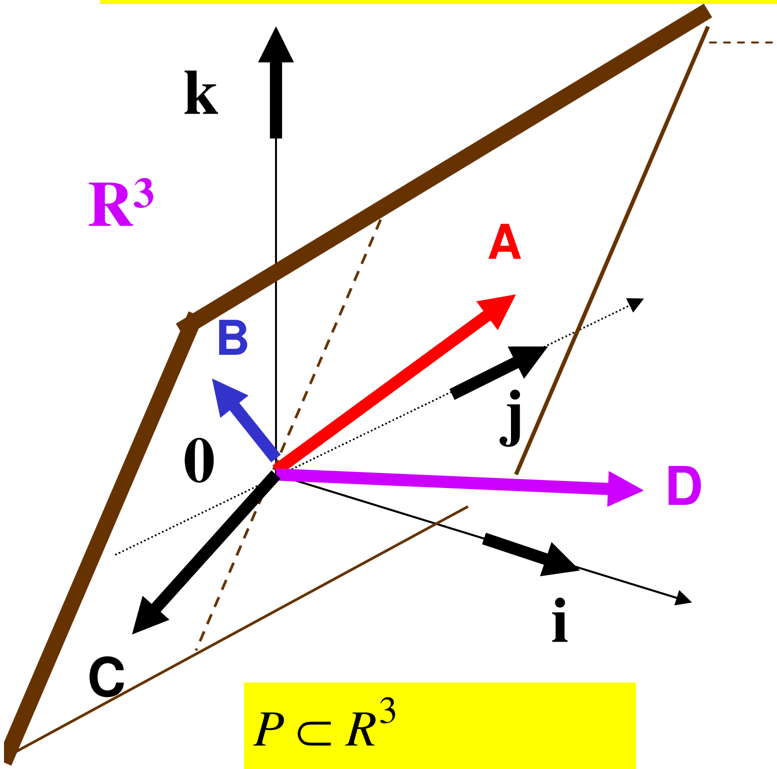
$Solving: \alpha_1 = 1, \qquad \alpha_2 = -2$

i.e. (1).A + (-2).B=C

or (1).A + (-2).B+(-1)C=0

Together, A, B and C are linearly dependent.

<u>Any</u> vector on this plane P can be expressed as a linear combination of <u>any</u> two chosen linearly independent vectors (for example A and B).



Plane P

(two dimensional vector space spanned by two linearly independent vectors A and B. $C=\alpha_1 A + \alpha_2 B$, $A, B, C \in P$ **)**

Important: P should pass through the origin O of the Euclidean 3-dimensional mother space R^3, <u>so that P carries the null vector</u>.

Note: Vector D does not lie in this two-dimensional plane (vector space) P, but it lies in the higher dimensional Euclidean mother space R^3.

$P \subset R^3$

$(\mathbf{A},\mathbf{B},\mathbf{C}) \in P \subset R^3$

$\mathbf{D} \notin P, \qquad \mathbf{D} \in R^3$

P is a subspace of R^3

$$\boxed{P \subset R^3}$$

The dimension of a vector space/subspace is the number of linearly independent vectors needed to span it.

1.3 Linear Vector Space

A linear vector space *V* is a set of vectors that satisfy the following rules

(1) Any linear combination of vectors in the set V of vectors should yield another vector that belongs to the same vector space.

i.e. **A linear vector space is closed under linear combination**.

If *{u}* and *{v}* are any two vectors in the vector space *V,* then any linear combination of these should also lie in the V

$$\boxed{If \quad \{u\},\{v\} \in V \quad then \quad \{w\} = c_1\{u\} + c_2\{v\} \quad \{w\} \in V}$$

where $c_1, c_2 \in R$. Here R is the set of real numbers.

(2) There exists a null vector {0} as a member of the vector space V, satisfying the following identity for all $\{u\} \in V$.

$$\boxed{\{u\} + \{0\} = \{u\}}$$

(3) The dimensions of a Vector Space V is the number of linearly independent vectors in it. The linearly independent vectors in the space V are called the **basis vectors**, spanning the space V.

(4) A *Hilbert space* is a one in which the **norm** (or **magnitude**) of any vector in it is always positive and is defined by the **inner product.** An inner product of any two vectors $\{a\}$ and $\{b\}$ in the Hilbert space is given by

$$\boxed{<a,b>=\{a\}^T[D]\{b\}} \qquad \textbf{(1.2)}$$

Here [D] is a symmetric positive definite matrix.

In Hilbert space the ***norm*** of any vector $\{a\}$ is given by

$$\boxed{\|a\|=\sqrt{<a,a>}} \qquad \textbf{(1.3)}$$

In the *n*-dimensional Euclidean space R^n, the ***norm*** of any vector **a** $(or\{\,a\})$ is given by the so-called **dot (scalar) product**

$$\boxed{\begin{array}{ll} <a,b>=\{a\}^T\{b\}=\mathbf{a}\cdot\mathbf{b} & [D]=[I] \\ \|a\|=\sqrt{<a,a>}=\sqrt{\{a\}^T\{a\}}=\sqrt{\mathbf{a}\cdot\mathbf{a}} & \end{array}}$$

Examples of vector spaces

Any "vector" in the real number line *R* can be a scalar multiple of 1 as basis, and it belongs to *R*.

Thus the space *R* is a one dimensional vector space.

The cartesian product set is a vector space R^2 which can be expressed as

$$R^2 = R \times R = \left(\{v\} : \{v\} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \quad \forall \quad \alpha_1, \alpha_2 \in R \right)$$

Basis vectors $[1, 0]^T$ and $[0, 1]^T$ span R^2. Dimension of R^2 is therefore 2.

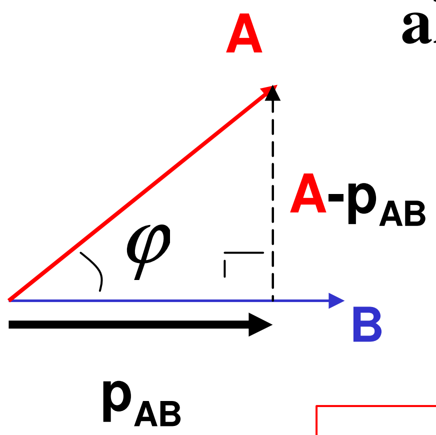
- Any vector $\{v\}$ in R^2 as a unique linear combination of these basis vectors as

$$\{v\} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \alpha_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \alpha_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

Likewise, any vector $\{v\}$ in the 3 dimensional vector space $R^3 = R \times R \times R$ can be expressed as a unique linear combination of 3 basis vectors spanning this vector space.

1.4 Projection of a vector along another vector

Scalar value of projection of a vector A along vector B



$$p_{AB} = \mathbf{A} \bullet \hat{\mathbf{b}}$$

$$\hat{\mathbf{b}} = \frac{\mathbf{B}}{B}$$ is the unit vector along **B**

Vector projection of a vector A along vector B

$$\mathbf{p}_{AB} = \left(\mathbf{A} \bullet \hat{\mathbf{b}}\right)\hat{\mathbf{b}} = \left(\mathbf{A} \bullet \frac{\mathbf{B}}{B}\right)\frac{\mathbf{B}}{B} = \left(\frac{\mathbf{A} \bullet \mathbf{B}}{\mathbf{B} \bullet \mathbf{B}}\right)\mathbf{B} \qquad \textbf{(1.4)}$$

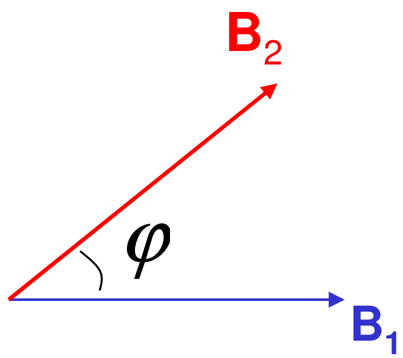
Orthogonality conditions satisfied by the projection

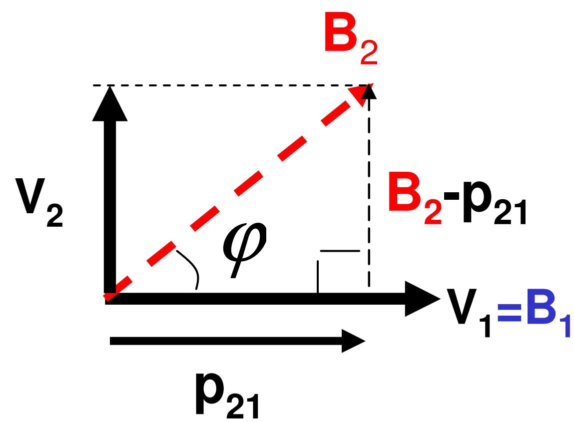
$$\mathbf{p}_{AB} \bullet \left(\mathbf{A} - \mathbf{p}_{AB}\right) = 0 \qquad \textbf{(1.5)}$$

$$\left\|\mathbf{A}\right\|^2 - \left\|\mathbf{p}_{AB}\right\|^2 = \left\|\left(\mathbf{A} - \mathbf{p}_{AB}\right)\right\|^2 \qquad \textbf{(1.6)}$$

1.5 Gram-Schmidt algorithm to find orthogonal basis set

Given two linearly independent vectors B_1 and B_2. Find a set of orthogonal vectors spanning the space in which B_1 and B_2 lies.





Orthogonality:

$$\mathbf{V}_2 \bullet \mathbf{V}_1 = 0$$

Let

$$\mathbf{V}_1 = \mathbf{B}_1$$

then

$$\mathbf{V}_2 = \mathbf{B}_2 - \mathbf{p}_{2,1} = \mathbf{B}_2 - \left(\frac{\mathbf{B}_2 \bullet \mathbf{B}_1}{\mathbf{B}_1 \bullet \mathbf{B}_1}\right)\mathbf{B}_1$$

Any vector W in space spanned by B_1 and B_2

$W = \alpha_1 B_1 + \alpha_2 B_2$

or

$W = \beta_1 V_1 + \beta_2 V_2$

$$\mathbf{V}_2 \bullet \mathbf{V}_1 = 0$$

Basis vectors are never unique, and can be arbitrarily scaled.

Gram-Schmidt algorithm to find orthogonal basis set spanning an *n*-dimensional vector subspace.

Given *n* linearly independent basis vectors $\{b_1\}, \{b_2\}, \ldots .\{b_n\}$ spanning the *n*-dimensional subspace V.

Find a set of orthogonal vectors spanning this space.

Any vector *w* in the *n*-dimensional subspace V can be expressed as

$$w = \sum \alpha_i \{b_i\} = \sum \beta_i \{v_i\} \qquad <v_i, v_j> = 0 \ \text{ for } i \neq j$$

Gram-Schmidt Algorithm to determine *n* numbers of orthogonal basis vectors spanning the n-dimensional vector space V in which the vectors $\{b_1\}, \{b_2\}, \ldots .\{b_n\}$ lie.

$$\{v_1\} = \{b_1\}$$

$$\{v_2\} = \{b_2\} - \frac{<b_2, v_1>}{<v_1, v_1>}\{v_1\}$$

$$\{v_k\} = \{b_k\} - \sum_{j=1}^{k-1} \frac{<b_k, v_j>}{<v_j, v_j>}\{v_j\} \qquad (1.7)$$

Basis vectors are never unique, and can be arbitrarily scaled.

Example 1. **Find an orthogonal basis set spanning the vector subspace V of the four-dimensional space R^4, given that the following basis set B spans subspace V.**

$$B = Span(V) = (\{b_1\},\{b_2\},\{b_3\}) \qquad where$$

$$\{b_1\} = [2 \quad 1 \quad 2 \quad 1]^T; \qquad \{b_2\} = [2 \quad 2 \quad 1 \quad 0]^T \quad ; \qquad \{b_3\} = [1 \quad 2 \quad 1 \quad 0]^T$$

Using the Gram-Schmidt algorithm

$$\boxed{\{v_1\} = \{b_1\} = [2 \quad 1 \quad 2 \quad 1]^T}$$

$$\{v_2\} = \{b_2\} - \frac{< b_2, v_1 >}{< v_1, v_1 >}\{v_1\} = [2 \quad 2 \quad 1 \quad 0]^T - \frac{4+2+2+0}{4+1+4+1}\{v_1\} = \frac{1}{5}[2 \quad 6 \quad -3 \quad -4]^T$$

Normalizing $\boxed{\{v_2\} = [2 \quad 6 \quad -3 \quad -4]^T}$

$$\{v_3\} = \{b_3\} - \frac{< b_3, v_1 >}{< v_1, v_1 >}\{v_1\} - \frac{< b_3, v_2 >}{< v_2, v_2 >}\{v_2\}$$

$$= [1 \quad 2 \quad 1 \quad 0]^T - \frac{2+2+2+0}{4+1+4+1}[2 \quad 1 \quad 2 \quad 1]^T - \frac{2+12-3+0}{4+36+9+16}[2 \quad 6 \quad -3 \quad -4]^T$$

$$= \frac{1}{13}[-7 \quad 5 \quad 4 \quad 1]^T$$

Normalizing $\boxed{\{v_3\} = [-7 \quad 5 \quad 4 \quad 1]^T}$

[$\{v_1\}$,$\{v_2\}$,$\{v_3\}$] is the orthogonal basis set spanning the three dimensional subspace V of the mother space R^4, $\boxed{V \subset R^4}$

$<v_i,v_j>=0$ for $i \neq j$

1.6 Orthogonal Projection of a vector in 3D space onto a 2D subspace (a plane)

Consider a plane P spanned by two orthogonal vectors u and v

Projection Formula for a vector A as $\overline{\mathbf{A}}$ onto the P subspace

$$\boxed{\overline{\mathbf{A}}=\left(\frac{\mathbf{A}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}+\left(\frac{\mathbf{A}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}\quad;\quad \mathbf{u}\cdot\mathbf{v}=0\quad;\quad \mathbf{u},\mathbf{v}\in P} \qquad (1.8)$$

$\overline{\mathbf{A}}$ is the orthogonal projection of A onto space P

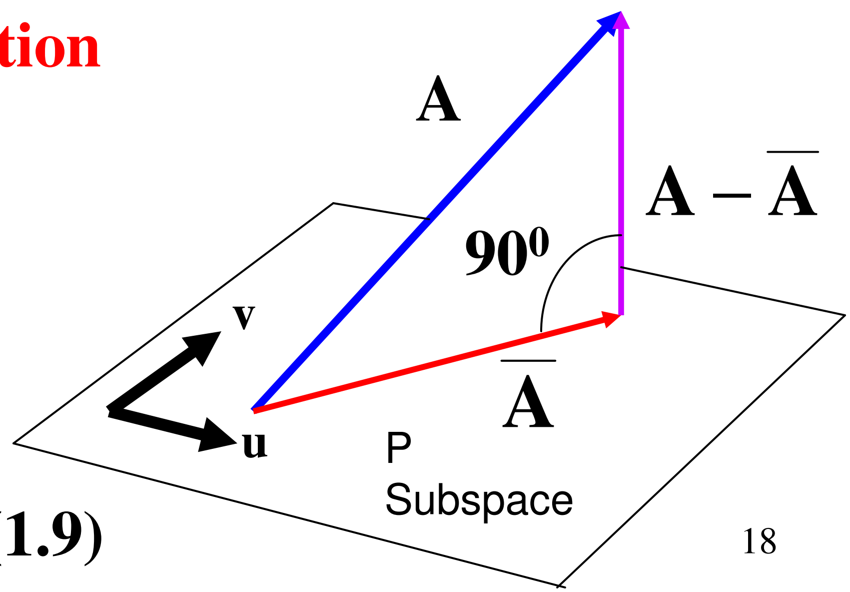
Properties of orthogonal projection

$$\left(\mathbf{A}-\overline{\mathbf{A}}\right)\bullet\mathbf{p}=0\quad;\quad \mathbf{p}\in P$$

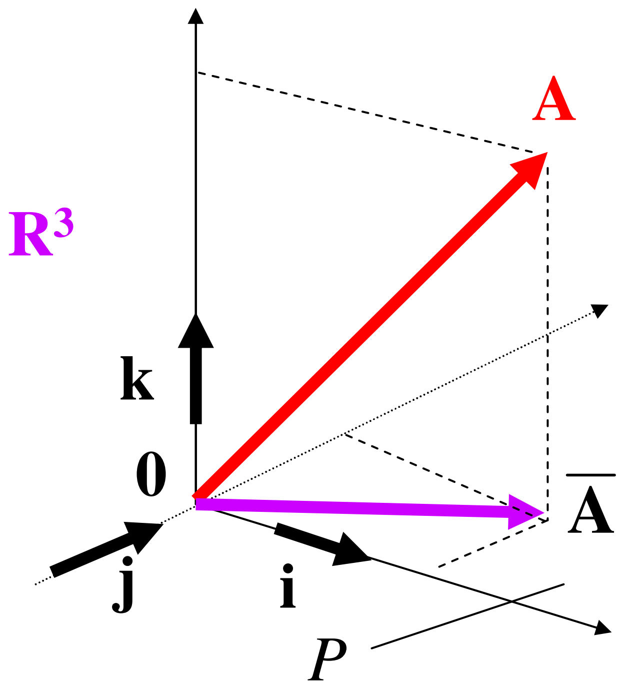
$$\left(\mathbf{A}-\overline{\mathbf{A}}\right)\bullet\overline{\mathbf{A}}=0\quad;\quad \overline{\mathbf{A}}\in P$$

$$\left|\mathbf{A}-\overline{\mathbf{A}}\right|^2=\left|\mathbf{A}\right|^2-\left|\overline{\mathbf{A}}\right|^2$$

(1.9)



1.7 Vector components as orthogonal projections



$$\mathbf{A} = \mathbf{i}A_1 + \mathbf{j}A_2 + \mathbf{k}A_3 \qquad \mathbf{A} \in R^3$$

$\overline{\mathbf{A}}$ is the Orthogonal Projection of vector A onto the plane (subspace) *P* spanned by orthogonal basis vectors i and j.

$$\overline{\mathbf{A}} = \mathbf{i}A_1 + \mathbf{j}A_2 \qquad \overline{\mathbf{A}} \in P$$

Vector A can also be represented as a linear combination of its orthogonal projections on the reference axes.

$$\mathbf{A} = \mathbf{i}A_1 + \mathbf{j}A_2 + \mathbf{k}A_3 = (\mathbf{A}\cdot\mathbf{i})\mathbf{i} + (\mathbf{A}\cdot\mathbf{j})\mathbf{j} + (\mathbf{A}\cdot\mathbf{k})\mathbf{k} \tag{1.10}$$

1.8 Orthogonal Projection of a vector in *n*-D space onto a lower dimensional, *m*-D subspace ($m<n$)

Consider a *m*-dimensional subspace P spanned by *m* numbers of orthogonal vectors $\{v_1\}$, $\{v_2\}$…$\{v_m\}$)

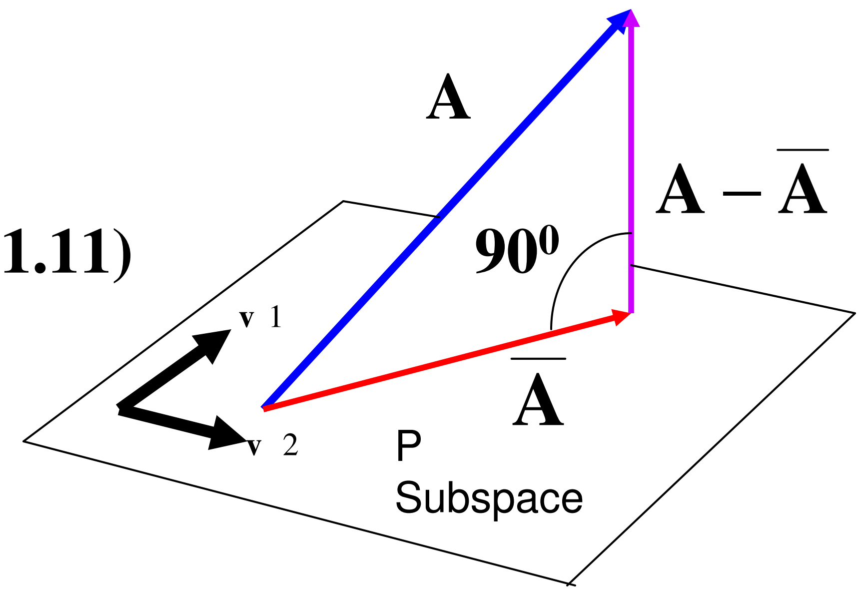
Projection Formula for a vector A existing in a higher n-dimensional subspace as $\overline{\mathbf{A}}$ onto the P subspace

$$\overline{\mathbf{A}} = \sum_{i=1}^{m} \left(\frac{\mathbf{A} \bullet \mathbf{v}_i}{\mathbf{v}_i \bullet \mathbf{v}_i} \right) \mathbf{v}_i \qquad \mathbf{v}_i \bullet \mathbf{v}_j = 0 \quad for \quad i \neq j$$

$$\left\{\overline{A}\right\} = \sum_{i=1}^{m} \frac{\{A\}^T \{v_i\}}{\{v_i\}^T \{v_i\}} \{v_i\} = \sum_{i=1}^{m} \frac{< A, v_i >}{< v_i, v_i >} \{v_i\}$$

$$< v_i, v_j >= 0 \quad for \quad i \neq j$$

(1.11)



Properties of orthogonal projection

$$< A - \overline{A}, p >= 0 \quad ; \quad \{p\} \in P$$

$$< \left(A - \overline{A} \right), \overline{A} >= 0$$

$$\left\| A \right\|^2 - \left\| \overline{A} \right\|^2 = \left\| A - \overline{A} \right\|^2$$

(1.12)

A simple geometric representation for visualizing projection in abstract high dimensional space.

Example 2. **Let V be a 2-dimensional subspace of 3-dmensional space R^3. The subspace V is spanned by the following given orthogonal basis vectors**

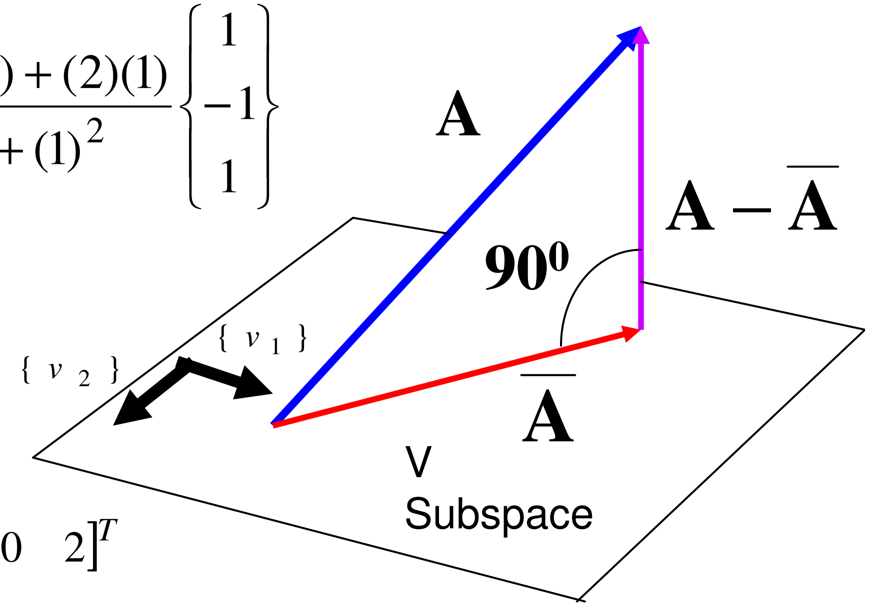
$$\{v_1\} = [1 \quad 2 \quad 1]^T, \qquad \{v_2\} = [1 \quad -1 \quad 1]^T$$

Find the orthogonal projection $\{\overline{A}\}$ **of the vector** $\{A\} = [-2 \quad 2 \quad 2]^T$ **onto the subspace V.**

$$\{\overline{A}\} = \frac{< A, v_1 >}{< v_1, v_1 >}\{v_1\} + \frac{< A, v_2 >}{< v_2, v_2 >}\{v_2\}$$

$$= \frac{(-2)(1)+(2)(2)+(2)(1)}{(1)^2+(2)^2+(1)^2}\begin{Bmatrix}1\\2\\1\end{Bmatrix} + \frac{(-2)(1)+(2)(-1)+(2)(1)}{(1)^2+(-1)^2+(1)^2}\begin{Bmatrix}1\\-1\\1\end{Bmatrix}$$

$$= \frac{4}{6}\begin{Bmatrix}1\\2\\1\end{Bmatrix} + \frac{(-2)}{3}\begin{Bmatrix}1\\-1\\1\end{Bmatrix} = \begin{Bmatrix}0\\2\\0\end{Bmatrix} = [0 \quad 2 \quad 0]^T$$



$$\{A-\overline{A}\} = \{A\} - \{\overline{A}\} = [-2 \quad 2 \quad 2]^T - [0 \quad 2 \quad 0]^T = [-2 \quad 0 \quad 2]^T$$

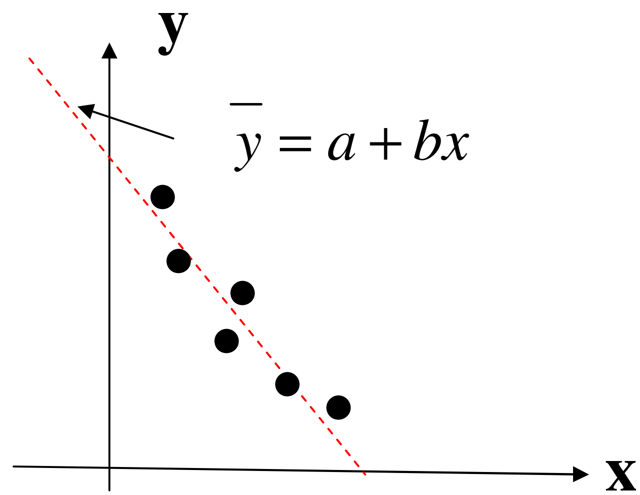
$$\|A-\overline{A}\|^2 = (-2)^2 + (0)^2 + (2)^2 = 8,$$

$$\|A\|^2 - \|\overline{A}\|^2 = \left[(-2)^2+(2)^2+(2)^2\right] - \left[(0)^2+(2)^2+(0)^2\right] = 12 - 4 = 8$$

1.9 Best-fits as Orthogonal Projections

We are given a scatter diagram of points (x_1, y_1), (x_2, y_2).. (x_n, y_n) etc., and asked to fit a best-fit straight line

$$\bar{y} = a + bx$$



Using the points (x_1, y_1), (x_2, y_2).. (x_n, y_n)

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ & \\ & \\ 1 & x_n \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \\ \\ y_n \end{Bmatrix}$$

$$[B]\{\alpha\} = \{y\} \qquad (1.13)$$

This is an inconsistent system of equations, since we have more Equations than the number of unknowns (a,b)

$$Pre-multiply \quad [B]\{\alpha\} = \{y\} \quad by \quad [B]^T \Rightarrow \boxed{[B]^T [B]\{\alpha\} = [B]^T \{y\}} \qquad (1.14)$$

This is the Normal Equation for best-fit

Solving the Normal Equation (1.9) we can get the best-fit straight line $\bar{y} = a + bx$

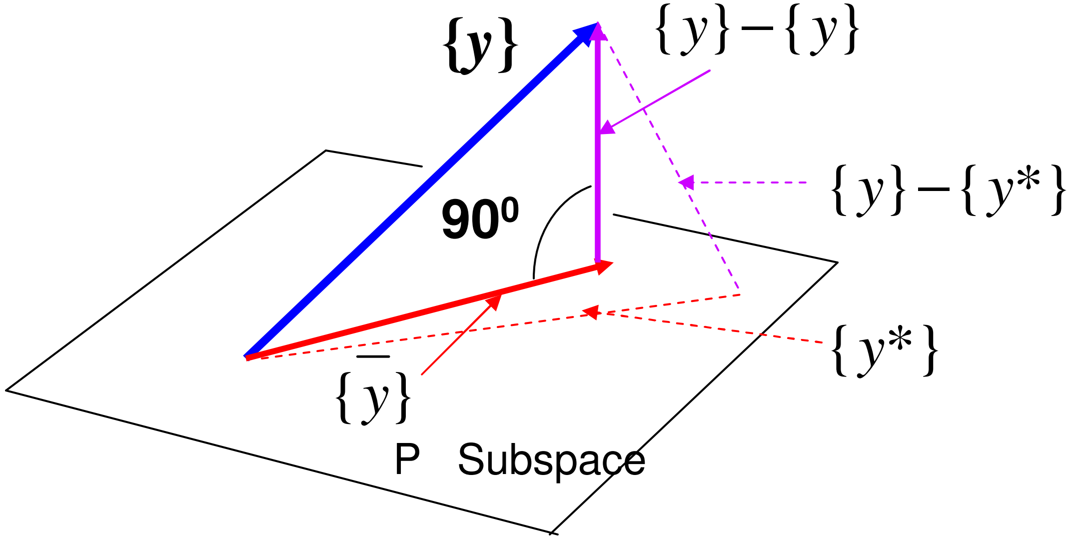
$$\bar{y} = a + bx = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix}$$

$$\bar{y} = [B(x)][\alpha]$$

$$[B]^T(\{y\} - [B]\{\alpha\}) = 0$$

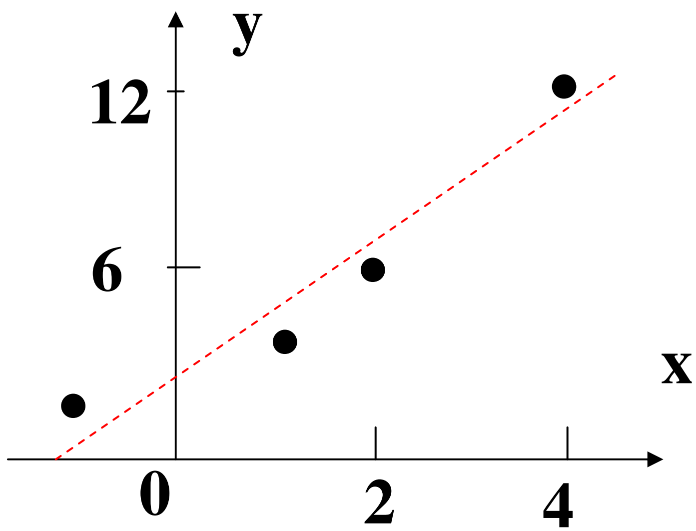
$$\{\bar{y}\}^T \{\{y\} - \{\bar{y}\}\} = 0 \qquad \textbf{(1.15)}$$

Error $\{y\} - \{\bar{y}\}$ **is orthogonal to the best fit** $\{\bar{y}\}$



Example 3. **Find the best-fit straight line to the following data**

x	-1	1	2	4
y	2	4	6	12



Let $\bar{y} = a + bx = \begin{bmatrix} 1 & x \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix}$

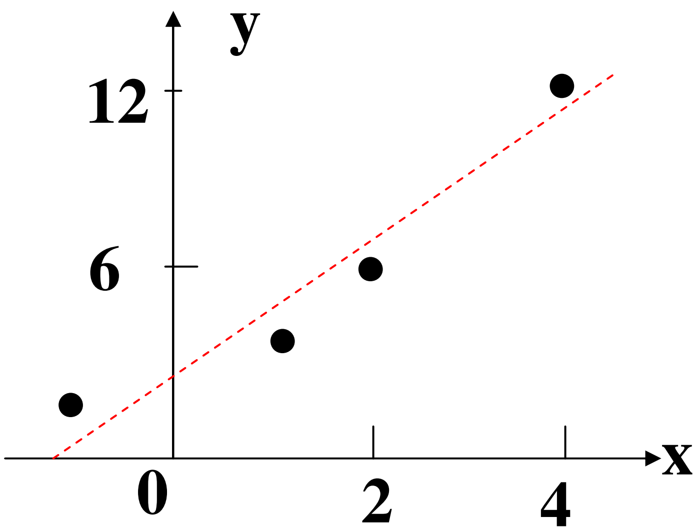
$$\bar{y} = [B(x)][\alpha]$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4 \\ 6 \\ 12 \end{Bmatrix}$$

$$\Rightarrow [B]\{\alpha\} = \{y\}$$

We have to find the linear best-fit solution to this inconsistent system of equations

x	-1	1	2	4
y	2	4	6	12

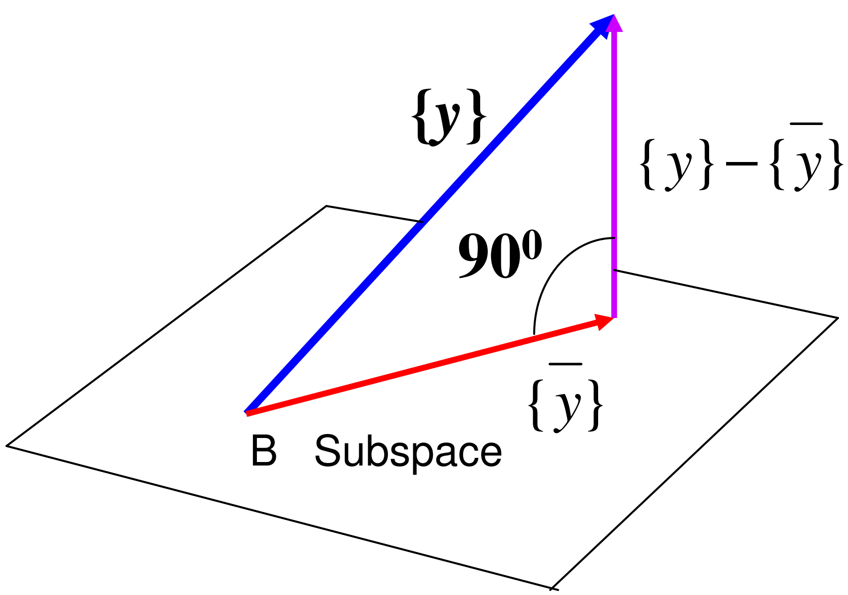


Method 1: Using Normal equation

$$\boxed{[B]^T [B]\{\alpha\} = [B]^T \{y\}}$$

$$\begin{bmatrix} 4 & 6 \\ 6 & 22 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 24 \\ 62 \end{Bmatrix}$$

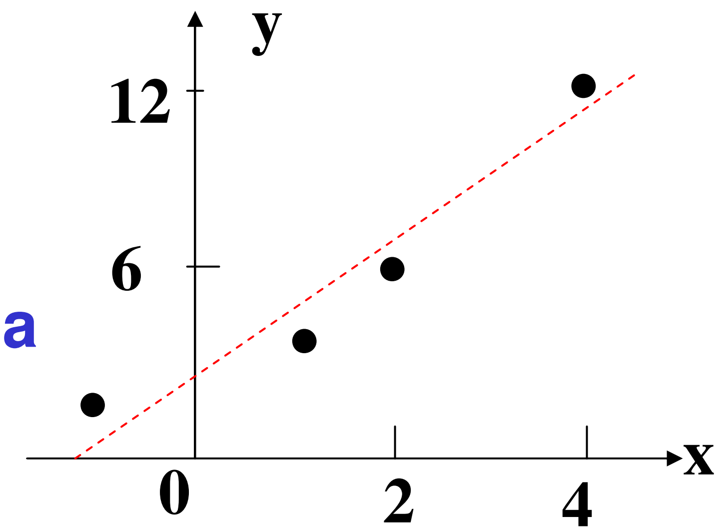
$$Solving: \quad a = 3, \quad b = 2$$



Thus the best-fit straight line is

$$\boxed{\bar{y} = 3 + 2x}$$

x	-1	1	2	4
y	2	4	6	12



Method 2: Using projection formula

$$[B]=\begin{bmatrix}1 & -1\\ 1 & 1\\ 1 & 2\\ 1 & 4\end{bmatrix}$$

Two-linearly independent vectors that lie in (and span) the space B are

$$\{b_1\}=\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T, \qquad \{b_2\}=\begin{bmatrix}-1 & 1 & 2 & 4\end{bmatrix}^T$$

Using the Gram-Schmidt algorithm, the orthogonal basis vectors spanning B subspace are to be established

$$\boxed{\{v_1\}=\{b_1\}=\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T}$$

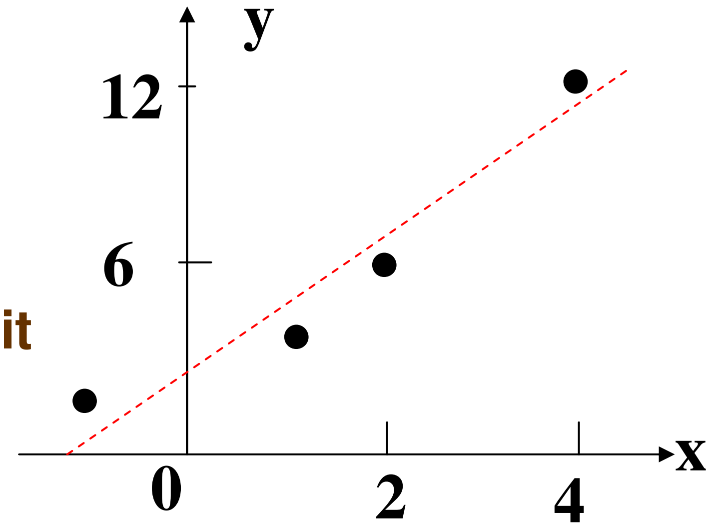
$$\{v_2\}=\{b_2\}-\frac{< b_2, v_1 >}{< v_1, v_1 >}\{v_1\}=\begin{bmatrix}-1 & 1 & 2 & 4\end{bmatrix}^T-\frac{6}{4}\{v_1\}=\frac{1}{2}\begin{bmatrix}-5 & -1 & 1 & 5\end{bmatrix}^T$$

Normalizing

$$\boxed{\{v_2\}=\begin{bmatrix}-5 & -1 & 1 & 5\end{bmatrix}^T}$$

$$\boxed{< v_1, v_2 >= 0}$$

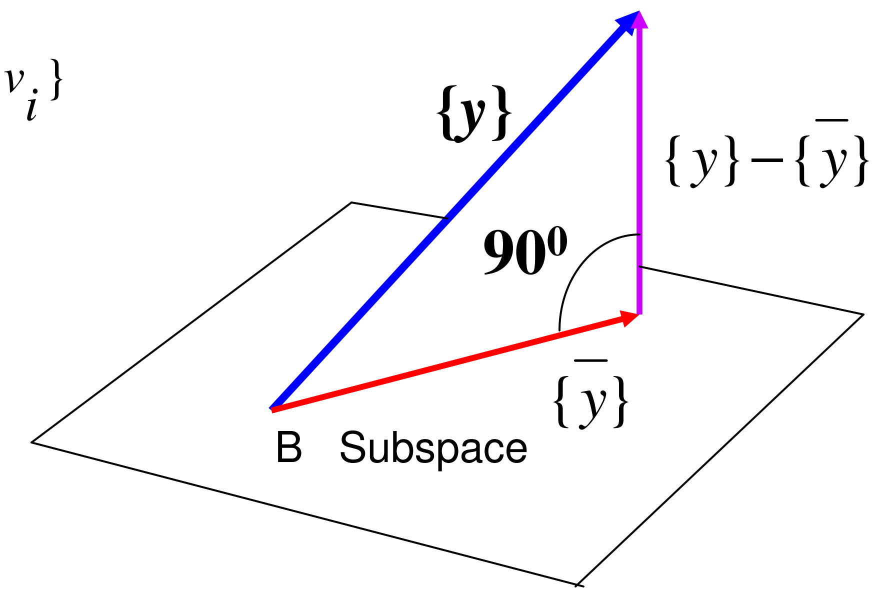
x	-1	1	2	4
y	2	4	6	12



Using projection formula, we find the best fit of the vector {y}=[2,4,6,8]^T

$$\{\bar{y}\} = \sum_{i=1}^{m} \frac{\{y\}^T \{v_i\}}{\{v_i\}^T \{v_i\}} \{v_i\} = \sum_{i=1}^{m} \frac{< y, v_i >}{< v_i, v_i >} \{v_i\}$$

$$\{\bar{y}\} = \frac{24}{4}\begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix} + \frac{52}{52}\begin{Bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 5 \\ 7 \\ 11 \end{Bmatrix}$$



$$\boxed{\bar{y} = 3 + 2x}$$ $\rightarrow$ **Orthogonal projection of the vector {y} onto the subspace B**

1.10 The Polynomial Vector (Function) Space

Consider a Vector Function Space of r-rowed vectors which are polynomial functions of degree (n-1) of some independent variable ξ, bounded by *–1* and *+1.*

Definition:

$$\boxed{P_n^r(\xi) = \left\{ \{p\} : \{p\} = \sum_{i=1}^{n} \{a\}_i \xi^{i-1}, \quad -1 \le \xi \le 1, \quad \{a\}_i \in R^r \right\}} \qquad \textbf{(1.16)}$$

Dimension of this space:

$$\boxed{\dim[P_n^r(\xi)] = r \times n} \qquad \textbf{(1.17)}$$

A Polynomial Function ***{p(ξ)}*** of degree ***(n-1)*** is a one-row vector ***(r=1)*** which is an element of the space

$$P_n(\xi) = P_n^{r=1}(\xi) = \left\{ \{p\} : \{p\} = \sum_{i=1}^{n} \{a\}_i \xi^{i-1}, \quad -1 \le \xi \le 1, \quad \{a\}_i \in R \right\}$$

$$\dim[P_n^{r=1}(\xi)] = n$$

1.11 The Legendre Polynomials

- **The *n*-dimensional function space $P_n^{r=1}(\xi)$ must be spanned by *n*-linearly independent vectors.**

The *n*- numbers orthogonal basis vectors spanning $P_n^{r=1}(\xi)$ are known as the Legendre Polynomials.

Example 4. A polynomial of degree 3 $\boxed{p_4 = a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3}$

belongs to the function space $P_4^{r=1}(\xi)$

The basis vectors spanning $P_4^{r=1}(\xi)$

$$b_1 = 1, \qquad b_2 = \xi, \quad b_3 = \xi^2, \quad b_4 = \xi^3$$

Inner product definition:

$$\boxed{< a, b >= \int_{-1}^{1} \{a\}^T \{b\}.d\xi \qquad a, b \in P_4^{\,1}(\xi)} \qquad \textbf{(1.18)}$$

Using the Gram-Schmidt algorithm, the orthogonal basis vectors spanning subspace $P_4^{r=1}(\xi)$ are to be established

$$\boxed{P_1 = b_1 = 1}$$

$$P_2 = b_2 - \frac{< b_2, P_1 >}{< P_1, P_1 >}\{P_1\} = \xi - \frac{< \xi, 1 >}{< 1,1 >} 1 = \xi - \frac{\int_{-1}^{1} \xi .1. d\xi}{\int_{-1}^{1} d\xi} 1 = \xi$$

$$\boxed{P_2 = \xi}$$

$$P_3 = b_3 - \frac{< b_3, P_1 >}{< P_1, P_1 >}\{P_1\} - \frac{< b_3, P_2 >}{< P_2, P_2 >}\{P_2\}$$

$$\boxed{P_3 = 3\xi^2 - 1}$$

$$P_4 = b_4 - \frac{< b_4, P_1 >}{< P_1, P_1 >}\{P_1\} - \frac{< b_4, P_2 >}{< P_2, P_2 >}\{P_2\} - \frac{< b_4, P_3 >}{< P_3, P_3 >}\{P_3\}$$

$$\boxed{P_4 = 5\xi^3 - 3\xi}$$

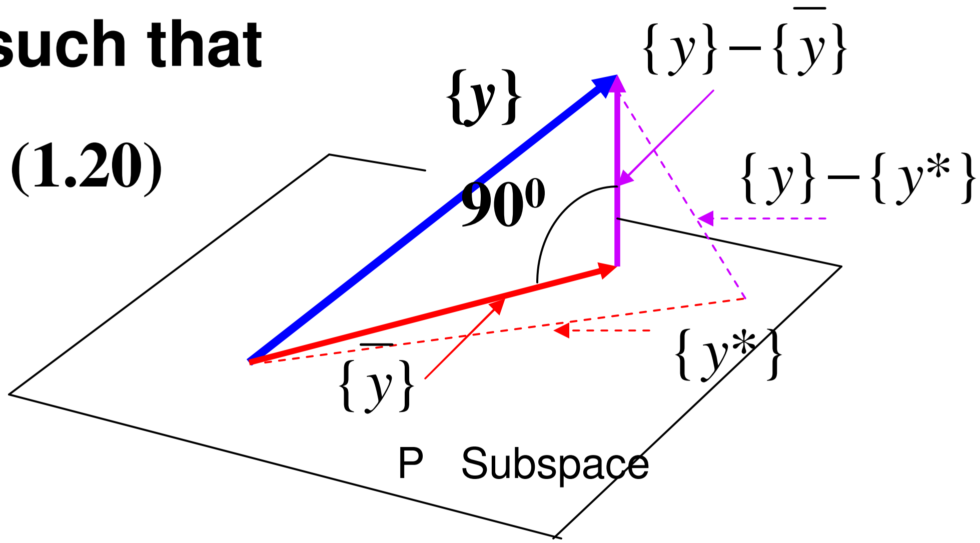
$$\boxed{\begin{gathered} span[P_4^1(\xi)] = [P_1, P_2, P_3, P_4] \\ < P_i, P_j >= \int_{-1}^{1} P_i P_j d\xi = 0, \qquad i \neq j \end{gathered}} \qquad \textbf{(1.19)}$$

1.12 Orthogonal Projection (Best-fits) of Polynomial Functions

If we seek a least square error solution of an original polynomial function $\{y\}$ then we actually look for a solution $\{\bar{y}\}$ such that

$$< \bar{y}, y - \bar{y} >= 0 \qquad (1.20)$$

(NORMAL EQUATION)



The minimum squared error is obtained when the error $\{\{y\}-\{\bar{y}\}\}$ is orthogonal to the approximate (best-fit) solution $\{\bar{y}\}$ lying on the approximation subspace.

Example 5. **Best fit the 2 degree polynomial**

$$y = p_3 = 1 + 2\xi + \xi^2$$

by a straight line.

Solution:

The straight line best fit $\bar{y}$ **to the quadratic** $y = p_3$ **is the orthogonal projection of the quadratic function onto the polynomial space** $P_2^{r=1}(\xi)$

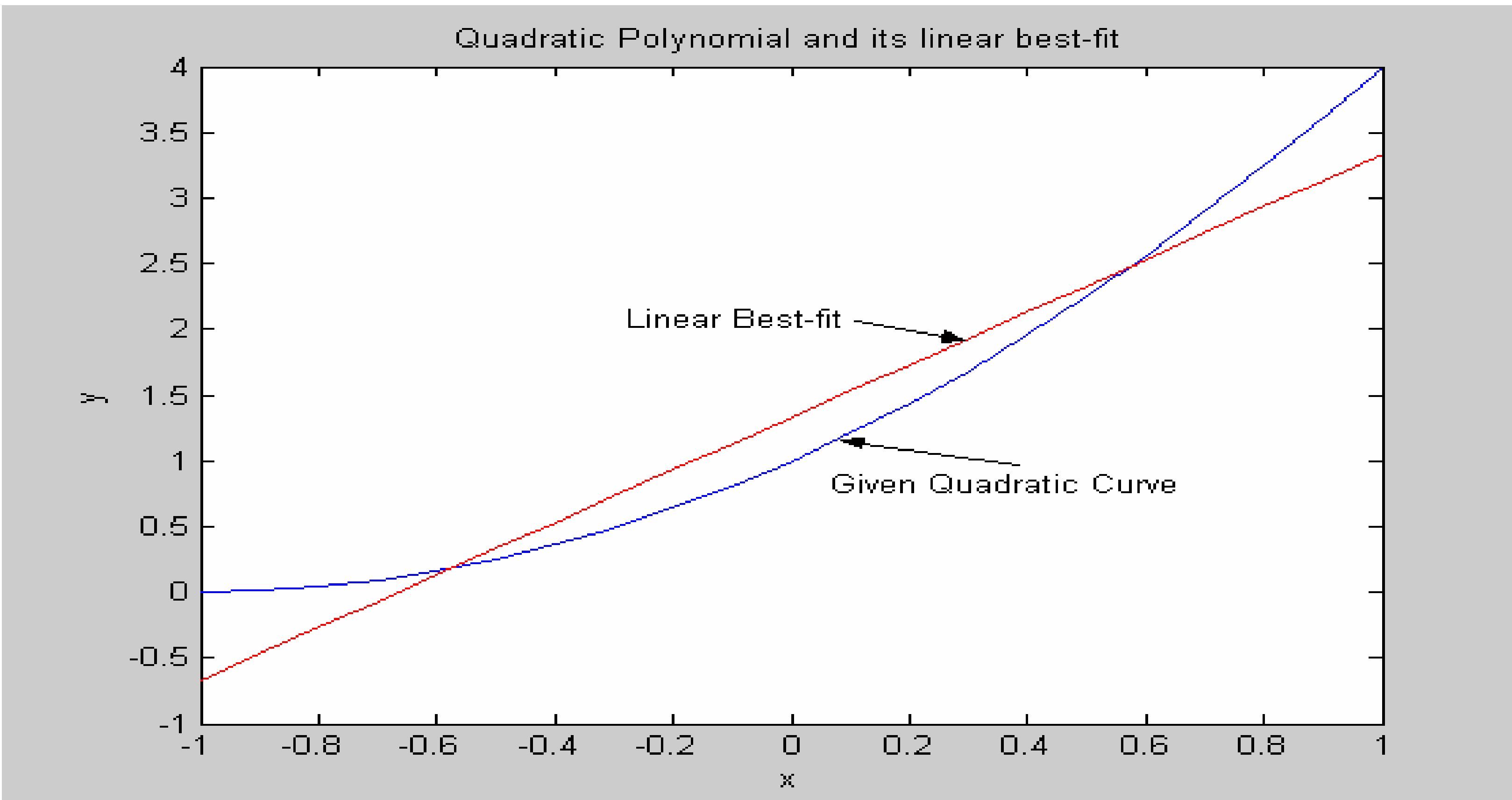
Using Projection Formula

$$\bar{y} = \sum_{i=1}^{2} \frac{< y, P_i >}{< P_i, P_i >} P_i = \frac{\int_{-1}^{1} y.1.d\xi}{\int_{-1}^{1} d\xi} + \frac{\int_{-1}^{1} y.\xi.d\xi}{\int_{-1}^{1} \xi^2 d\xi} \xi = \frac{4}{3} + 2\xi$$

Given Quadratic Curve is

$$y = p_3 = 1 + 2\xi + \xi^2 = \frac{4}{3}P_1 + 2P_2 + \frac{1}{3}P_3 \qquad -1 \le \xi \le 1$$

Linear Best-fit to y is $\bar{y} = \frac{4}{3} + 2\xi = \frac{4}{3} + 2P_2$

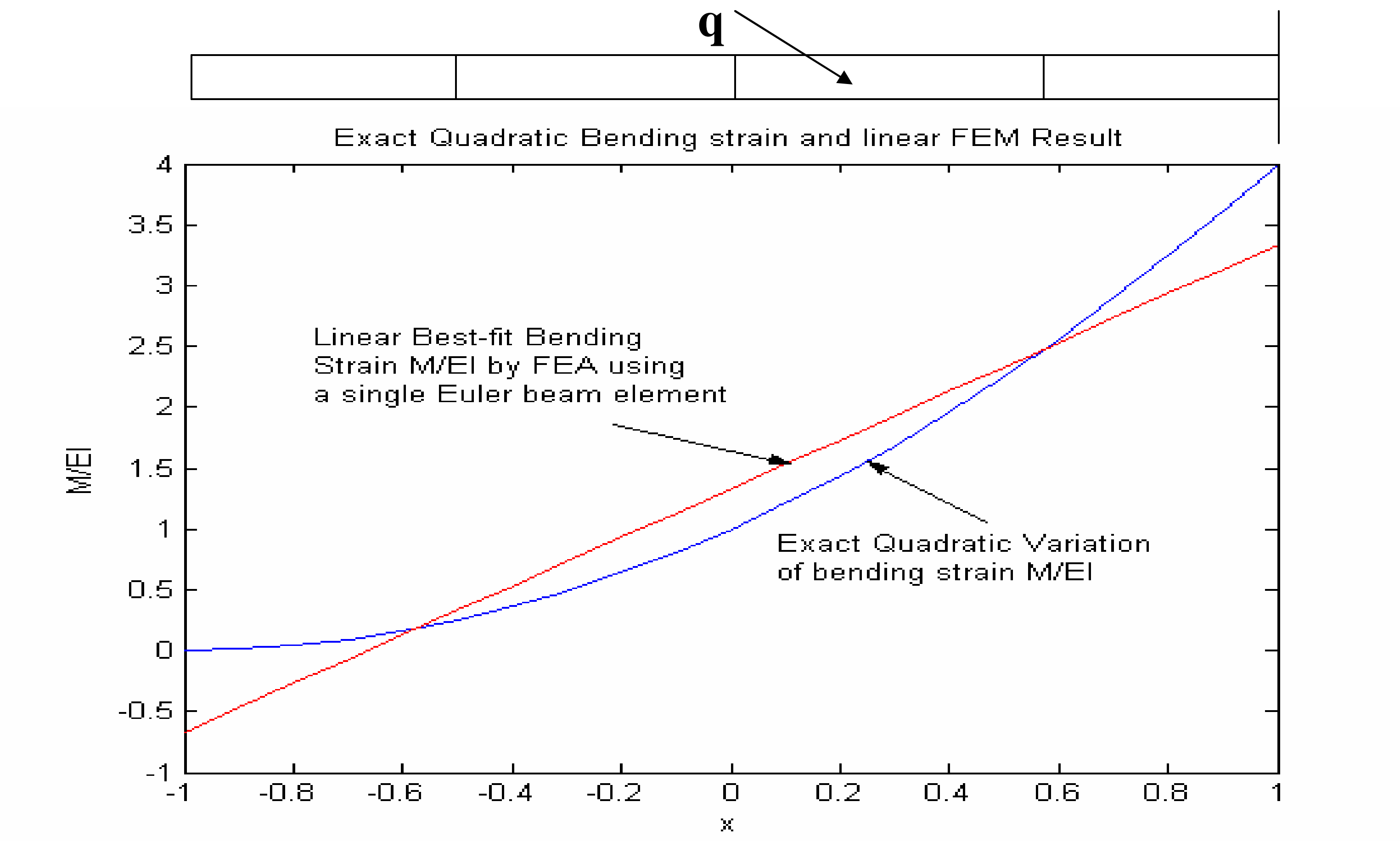


An indication of the best-fit rule in FEA …

CANTILEVER BEAM ANALYSIS USING A SINGLE EULER BEAM ELEMENT OF LENGTH L

Uniformly distributed loading is q per unit length.

q is such that fixed end curvature (bending strain) is $qL^2/2EI=4$ (m^{-1})



1.13 Fourier Series as orthogonal projection of any periodic function

A periodic function $f(t)$ of period 2π: $\boxed{f(t+2\pi) = f(t)}$

Fourier series

Definition Let f be a piecewise continuous function on $[-\pi, \pi]$. Then the **Fourier series** of f is the series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the coefficients a_n and b_n in this series are defined by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

and are called the **Fourier coefficients** of f.

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx=\frac{<f(x),1>}{<1,1>}1+\sum_{n=1}^{\infty}\frac{<f(x),\cos nx>}{<\cos nx,\cos nx>}\cos nx+\sum_{n=1}^{\infty}\frac{<f(x),\sin nx>}{<\sin nx,\sin nx>}\sin nx$$

$$<a,b>=\int_{-\pi}^{\pi}a.b.dx$$

$$\|a\|^2=<a,a>=\int_{-\pi}^{\pi}a.a.dx$$

$$a_0=\frac{<f(x),1>}{<1,1>}=\frac{\int_{-\pi}^{\pi}f(x)dx}{2\pi},\qquad a_n=\frac{<f(x),\cos nx>}{<\cos nx,\cos nx>}=\frac{\int_{-\pi}^{\pi}f(x)\cos nxdx}{\pi},\qquad b_n=\frac{<f(x),\sin nx>}{<\sin nx,\sin nx>}=\frac{\int_{-\pi}^{\pi}f(x)\sin nxdx}{\pi}$$

In practice, we use a finite number (N) of terms;

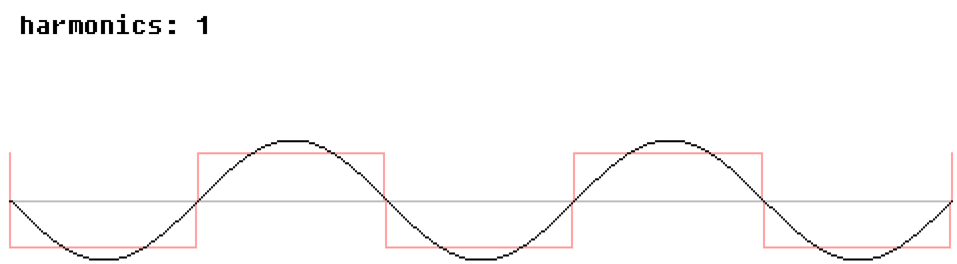
$$\overline{f}(x)=a_0+\sum_{n=1}^{N}a_n\cos nx+\sum_{n=1}^{N}b_n\sin nx=\frac{<f(x),1>}{<1,1>}1+\sum_{n=1}^{N}\frac{<f(x),\cos nx>}{<\cos nx,\cos nx>}\cos nx+\sum_{n=1}^{N}\frac{<f(x),\sin nx>}{<\sin nx,\sin nx>}\sin nx$$

Thus $\overline{f}(x)$ is the orthogonal projection of $f(x)$ onto a subspace of $2N+1$ dimensions, spanned by orthogonal basis vectors $\{1, \sin(nx), \cos(nx): n=1,2,\ldots N, \quad -\pi \le x \le \pi\}$

$$\|f\|^2-\|\overline{f}\|^2=\|f-\overline{f}\|^2$$

Using Fourier series we can write an ideal square wave as an infinite series of the form

$$
\begin{aligned}
x_{\text{square}}(t) &= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin\left((2k-1)2\pi f t\right)}{(2k-1)} \\
&= \frac{4}{\pi}\left(\sin(2\pi f t) + \frac{1}{3}\sin(6\pi f t) + \frac{1}{5}\sin(10\pi f t) + \cdots\right)
\end{aligned}
$$



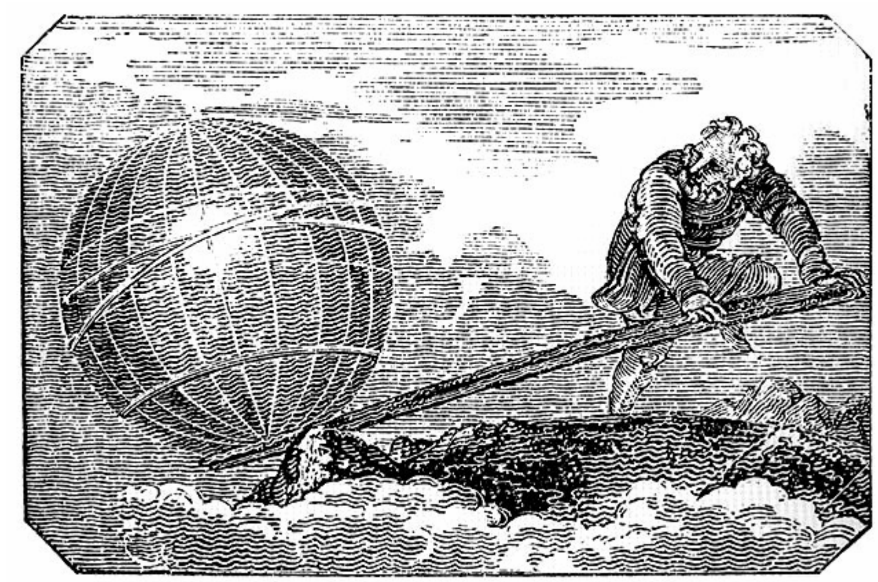
Lecture 3

FEA creates shadows…

Chapter 2

How the "Principle of Virtual Work" works to make FEA the best-fit

Archimedes was the first person to use the concept of **Virtual Work** in his calculations for the Lever.

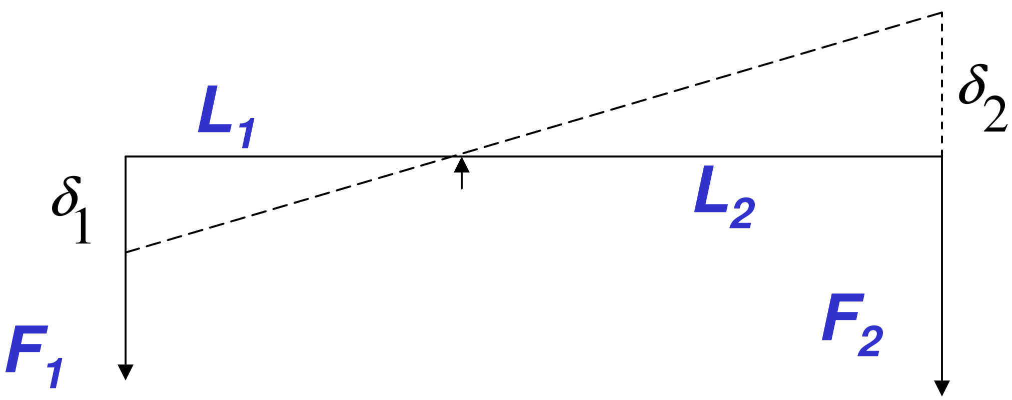


"Give me a place to stand on, and I will move the Earth."

Equilibrium: Virtual Work Principle

$$F_1\delta_1 - F_2\delta_2 = 0$$

Geometric Compatibility $\dfrac{\delta_1}{L_1} = \dfrac{\delta_2}{L_2}$



Together

$$F_1 L_1 = F_2 L_2$$

2.1 Application of the Principle of Virtual Work

2.1.1 The weak form at element level with exact solution u in the differential equation

- A linear differential equation of a conservative system (with $\mathcal{L}$ as a self-adjoint operator),

$$\mathcal{L}(u) = f \qquad (2.1)$$

- Weak form at the element level is obtained from applying the virtual work principle

$$\int_e u^h (\mathcal{L}u - f)dx = 0 \qquad (2.2)$$

This leads to the weak form as $$a(u^h, u)^e = (u^h, f) + \lfloor u^h, R^e \rfloor_B \qquad (2.3)$$

$a(u,u^h)^e$ is the **symmetric, bilinear functional** $$a(u,u^h)^e = a(u^h,u)^e \qquad (2.4)$$

$$(u^h, f) = \int_e u^h . f\, dx \qquad (2.5)$$

$\left[u^h, R^e\right]_B = \{u^h\}_B^T \{R^e\}$ = **Virtual Work done by the ANALYTICAL nodal reaction vector {R^e}** *from connectivity* at the **element boundary**

2.1.2 The weak form at element level with approximate solution for the differential operator (FEA form)

$$\int_e u^h(\mathcal{L}u^h - f)dx = a(u^h,u^h)^e - (u^h, f) - \left[u^h, Q^{he}\right]_B \neq 0 \qquad (2.6)$$

Here $\{Q^{he}\}$ **= FE computed Nodal Stress Resultant using** u^h

However, we may replace the approximate nodal stress resultant vector appropriate *nodal reaction vector for equilibrium*

$$a(u^h,u^h)^e = (u^h, f) + \left[u^h, R^{he}\right]_B \qquad (2.7)$$

$$R^{he} \neq Q^{he} \qquad (2.8)$$

Computed nodal reaction vector NOT EQUAL TO nodal stress resultant vector

With FE approximation, each element is only in external equilibrium, but not in internal equilibrium.

The FEA computed Nodal Reaction Vector for the element is

$$\{R^{he}\} = [K^e]\{\delta^e\} - \{F^e\} \qquad (2.9)$$

2.2 FEA Error Statement

Equation (2.3): $$a(u^h, u)^e = (u^h, f) + \lfloor u^h, R^e \rfloor_B$$

Equation (2.7):
(Actual FEA) $$a(u^h, u^h)^e = (u^h, f) + \lfloor u^h, R^{he} \rfloor_B$$

Subtracting equation (2.3) from (2.7)

$$\begin{aligned} a(u^h, u^h)^e - a(u^h, u)^e &= \lfloor u^h, R^{he} - R^e \rfloor_B \\ &= \{\delta^e\}^T \{R^{he} - R^e\} \end{aligned} \tag{2.10}$$

This equation governs all FEM errors in a very general sense

2.3 The Bilinear Symmetric Form and the inner product

- Element Strain Vector by FEA: $\{\varepsilon^h\} = [B]\{\delta^e\}$ (2.11)

- Analytical element strain vector : $\{\varepsilon\}$

- Bilinear form and Inner product definition :

$$a(u^h, u)^e = \int_e \{\varepsilon^h\}^T [D]\{\varepsilon\} dx = <\varepsilon^h, \varepsilon> \qquad (2.12)$$

$$a(u^h, u^h)^e = \int_e \{\varepsilon^h\}^T [D]\{\varepsilon^h\} dx = <\varepsilon^h, \varepsilon^h> = \|\varepsilon^h\|^2 \qquad (2.13)$$

Here [D] is the element rigidity matrix

2.4 The Best-Fit Paradigm in FEA

CASE A: Agreement of Nodal Reactions in Elements

- **In all statically determinate problems, it has been observed that the analytical nodal reactions are exactly reproduced by finite element computations, i.e.**

$$\{R^{he}\}=\{R^{e}\} \tag{2.14}$$

Then from equation (2.10), we have

$$a(u^h,u^h)^e = a(u^h,u)^e \tag{2.15}$$

We get **NORMAL EQUATION** (for Orthogonal Projection)

$$<\varepsilon^h,\varepsilon^h>=<\varepsilon^h,\varepsilon> \tag{2.16}$$

or

$$\boxed{<\varepsilon^h,\varepsilon-\varepsilon^h>=0} \tag{2.17}$$

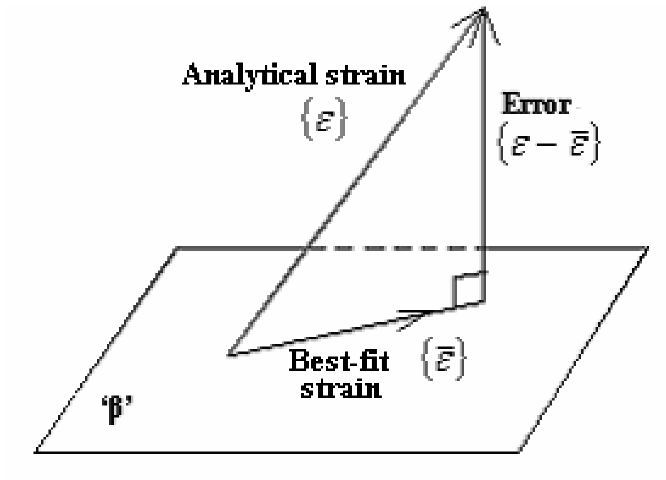
$$\boxed{\int_e [B]^T[D][B]\{\delta^e\}dx=\int_e [B]^T[D]\{\varepsilon\}dx} \tag{2.18}$$

Conclusion: Finite Element Strain $\{\varepsilon^h\}$ is actually an Orthogonal Projection (best-fit) of the analytical strain $\{\varepsilon\}$ onto the B Subspace

Using the *Gram-Schmidt process*, then orthogonal basis vectors spanning the m-dimensional space B can be found out. Hence by Projection Formula

$$\boxed{\{\varepsilon^h\}=\{\bar{\varepsilon}\}=\sum_{i=1}^{m}\frac{<\varepsilon,v_i>}{<v_i,v_i>}\{v_i\}, \quad <v_i,v_j>=0 \quad for \quad i\neq j} \tag{2.19}$$

FEA strain vector is the orthogonal projection of the analytical strain vector, onto the B Subspace.



The Error-Energy Rule (Pythagoras Theorem)

$$\left\|\varepsilon - \varepsilon^h\right\|^2 = \left\|\varepsilon\right\|^2 - \left\|\varepsilon^h\right\|^2 \qquad (2.20)$$

i.e.

The Energy of the Error= Error of the Energies

2.4 The Best-Fit Paradigm in FEA (..Continued)

CASE B: In many statically indeterminate problems, the FEM Computed Nodal Reactions in elements do not agree with the analytical ones.

$$\{R^{he}\} \neq \{R^{e}\}$$

- **This leads to a *prima facie* violation of the best-fit rule:**

$$\begin{aligned} a(u^h,u^h)^e - a(u^h,u)^e &= \lfloor u^h, R^{he} - R^e \rfloor_B \\ &= \left\{\delta^e\right\}^T \left\{R^{he} - R^e\right\} \\ &\neq 0 \end{aligned}$$

- **This means that the discretisation and approximation process of FEA have induced a stiffening force in the system from the error of the reaction**

$$\left\{R^{he} - R^e\right\}$$

- **However, note that if we consider the modified analytical solution for the stiffened system from the extraneous force from the reaction error, then the FEA results are actually the best-fit strains of the stiffened analytical solution.**

2.5 How the B Subspace (for strain projection) emerges

The strain displacement matrix [B] emerges from the FE formulation :

$$\{\varepsilon^h\} = [B]\{\delta^e\}$$

The B subspace is the vector function space in which all the column vectors of the [B] matrix lies. But these vectors in [B] need not be all linearly independent.

The B space is a subspace of the general Polynomial Space of *r*-rows.

$$B \subset P_n^r \quad ; \quad P_n^r = \left\{ \{p\} : \{p\} = \sum_{i=1}^{n} \{a_i\} \xi^{i-1}, -1 \le \xi \le 1 \,,\ \{a_i\} \in R^r \right\}$$

$$m = \dim(B) < \dim P_n^r = r \times n$$

Using Gram-Schmidt process, one can generate *m*-numbers of the *non-zero orthogonal basis vectors* spanning the *m-dimensional* B subspace.

m=dim(B)=Rank of the element stiffness matrix

If the element does not suffer from rank deficiency, then the dimension *m* of the B space is given by

dim(B)=N-R

N=Total number of degrees of freedom of the element

R=Total number of physical rigid body motions of the element

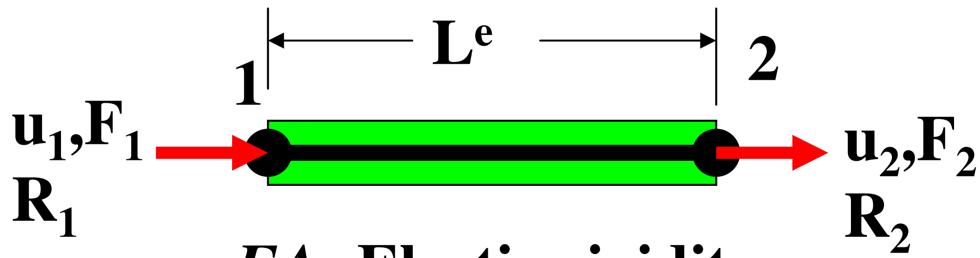
Lecture 3

The proof of the pudding is in the eating…

Chapter 3

Real Examples:
Simple one-dimensional elements

3.1 The Simple Bar Element



***EA*=Elastic rigidity**

Differential Equation:

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right)-q=0 \qquad (3.1)$$

Weak form (with exact solution for the differential equation):

$$\int_e\left[-\frac{d}{dx}\left(EA\frac{du}{dx}\right)-q\right].u^h.dx=0 \qquad (3.2)$$

Integration by parts,

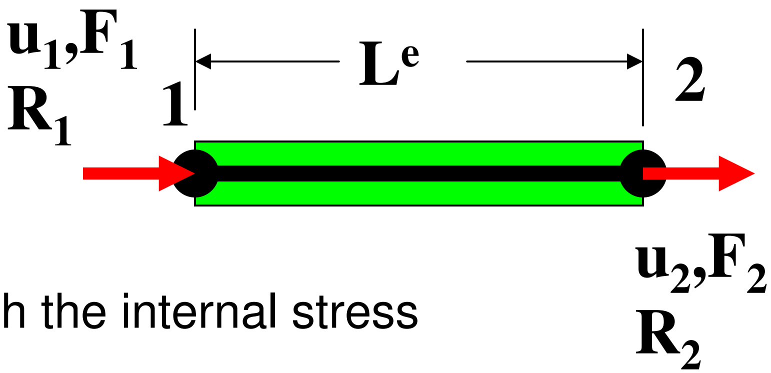
$$\int_e\frac{du^h}{dx}\left(EA\frac{du}{dx}\right)dx-\int_e u^h q dx-\left[u_1\left(-EA\frac{du}{dx}\right)_{x=x_1}+u_2\left(EA\frac{du}{dx}\right)_{x=x_2}\right]=0 \qquad (3.3)$$

- **The Weak form (not used in FEA) is**

$$\int_e \frac{du^h}{dx}\left(EA\frac{du}{dx}\right)dx = \int_e u^h q dx + \left[u_1\left(-EA\frac{du}{dx}\right)_{x=x_1} + u_2\left(EA\frac{du}{dx}\right)_{x=x_2} \right] \quad (3.4)$$

Here we have the general form

$$\boxed{a(u^h,u)^e = (u^h,q)^e + [u_1 R_1^e + u_2 R_2^e]} \quad (3.5)$$



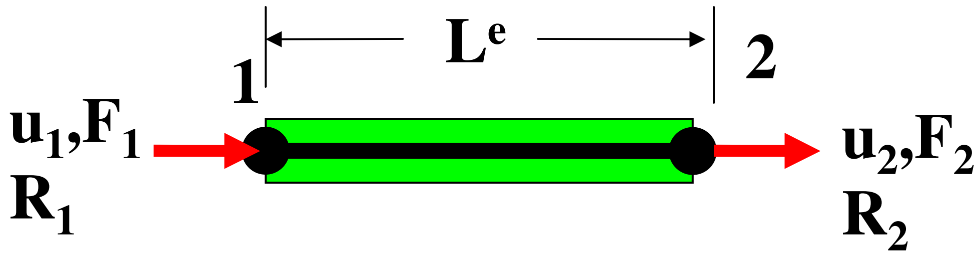
The Analytical Reactions at the nodes match the internal stress resultants

$$R_1^e = \left(-EA\frac{du}{dx}\right)_{x=x_1} \qquad R_2^e = \left(EA\frac{du}{dx}\right)_{x=x_2} \quad (3.6)$$

Equation (3.6) is satisfied only if the exact function u is used in the differential equation.

This means that with the exact function *u* in the differential equation, the element is in both *external* and *internal* equilibrium.

- **Weak form for the FEA:**



$$\int_e \left[-\frac{d}{dx}\left(EA\frac{du^h}{dx}\right) - q \right].u^h.dx$$
$$= a(u^h, u^h)^e - (u^h, q)^e - \left\{ u_1 \left(-EA\frac{du^h}{dx}\right)_{x=x_1} + u_2 \left(EA\frac{du^h}{dx}\right)_{x=x_2} \right\} \qquad (3.7)$$
$$\neq 0 \qquad !!$$

But with **proper nodal reactions** for only **external equilibrium** of the element (using this approximation),

$$a(u^h, u^h)^e - (u^h, q)^e - [u_1 R_1^{h,e} + u_2 R_2^{h,e}] = 0$$

or

$$\boxed{a(u^h, u^h)^e = (u^h, q)^e + [u_1 R_1^{h,e} + u_2 R_2^{h,e}]} \qquad (3.8)$$

Penalty for using approximate function is that the approximate internal stress resultants do not agree with the corresponding nodal reactions

$$R_1^{h,e} \neq \left(-EA\frac{du^h}{dx}\right)_{x=x_1} \quad and \quad R_2^{h,e} \neq \left(EA\frac{du^h}{dx}\right)_{x=x_2} \qquad (3.9)$$

Element is in external equilibrium; it is NOT in internal equilibrium (!)

3.2 The best-fit rule in simple bar element

The following relationship holds good in general for any element

$$a(u^h, u^h)^e - a(u^h, u)^e = \lfloor u^h, R^{he} - R^e \rfloor_B$$
$$= \{\delta^e\}^T \{R^{he} - R^e\}$$

For the bar element:

$$\int_e \left[\frac{du^h}{dx} \left(EA \frac{du^h}{dx} \right) \right] .dx - \int_e \left[\frac{du^h}{dx} \left(EA \frac{du}{dx} \right) \right] .dx = \{\delta^e\}^T \{R^{he} - R^e\} \qquad (3.10)$$

Here the Analytical Reaction Vector components are

$$R_1^e = \left(-EA \frac{du}{dx} \right)_{x=x_1} \qquad R_2^e = \left(EA \frac{du}{dx} \right)_{x=x_2}$$

The FEA computed Nodal Reaction Vector for an element is

$$\{R^{he}\} = \lfloor K^e \rfloor \{\delta^e\} - \{F^e\}$$

3.2.1 Case A: When the Analytical Reactions are conserved by FEA

- In this case $\{R^{he}\} = \{R^{e}\}$

- Hence we have the following as direct consequences,

$$a(u^h, u^h)^e = a(u^h, u)^e \qquad < \varepsilon^h, \varepsilon - \varepsilon^h >= 0$$

$$\int_e [B]^T [D][B]\{\delta^e\} dx = \int_e [B]^T [D]\{\varepsilon\} dx$$

- For the linear bar element (with linear Lagrangian shape functions)

$$\varepsilon^h = [B]\{\delta^e\} = \begin{bmatrix} -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad [B] = \begin{bmatrix} -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix}$$

$$N = Degrees \quad of \quad freedom = 2, \qquad R = Rigid \quad body \quad motion = 1$$

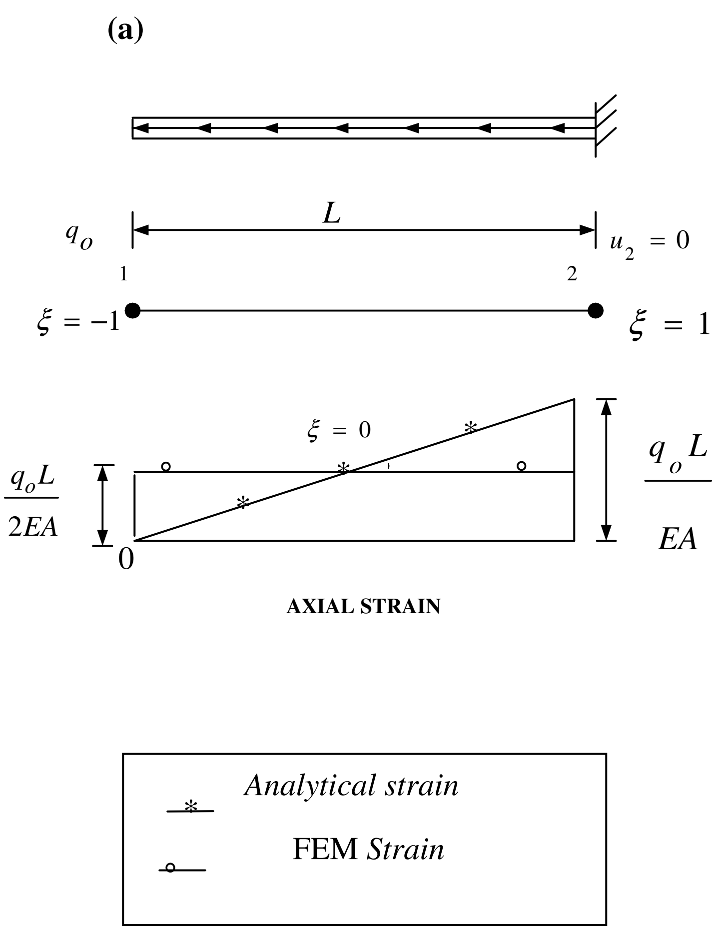
$$\dim(B) = 1, \qquad B \quad spanned \quad by \quad \nu = 1 \qquad (Applying \quad Gram - Schmidt)$$

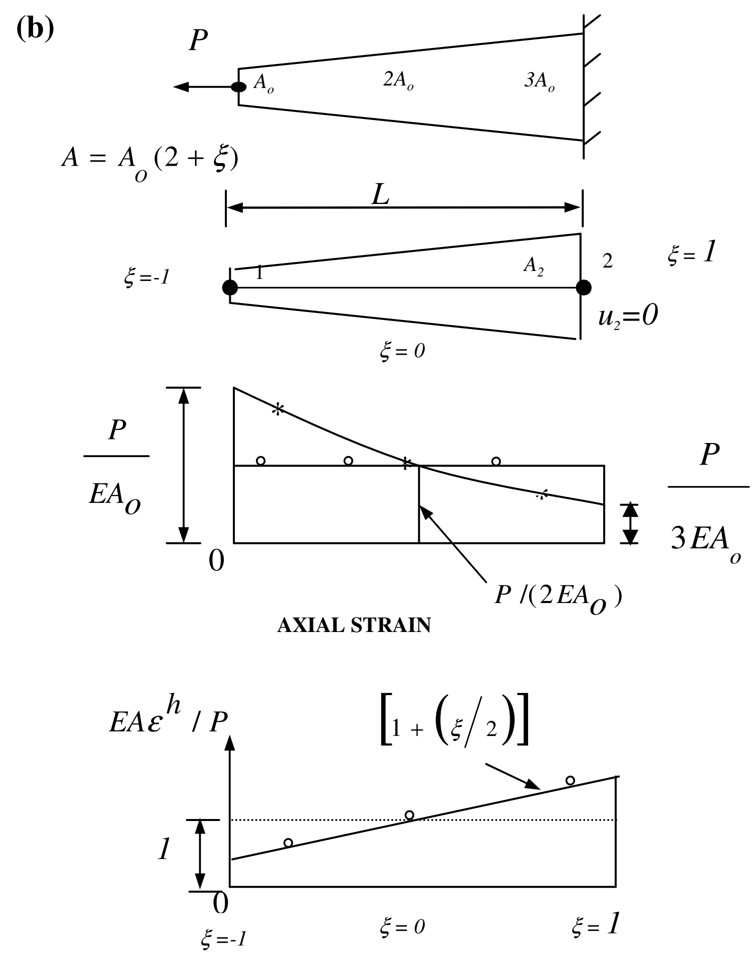
- FEA gives Best-fit strain $\{\varepsilon^h\} = \{\bar{\varepsilon}\} = \dfrac{< \varepsilon, 1 >}{< 1, 1 >} 1$

EXAMPLE 1. ***Cantilever bar analysis with a single linear bar element.***

	(a) Bar with constant section area A subjected to uniformly distribute axial load q_0.	***(b) Bar with varying sectional area ($A_1=A_o$, $A_2=3A_o$)subjected to axial tip load P.***
Rigidity matrix [D]	EA	$EA_0(2+\xi)$
Orthogonal basis vector spanning 1-D B subspace	$\{v\}_1=1$	$\{v\}_1=1$
Analytical strain vector $\{\varepsilon\}$	$\dfrac{q_0L(1+\xi)}{2EA}$	$\dfrac{P}{EA_0(2+\xi)}$
$\{\varepsilon^h\}=\{\bar{\varepsilon}\}=\dfrac{<\varepsilon,v_1>}{<v_1,v_1>}\{v_1\}$	$\dfrac{q_0L}{2EA}$	$\dfrac{P}{2EA_0}$
$\left\Vert\varepsilon-\varepsilon^h\right\Vert^2=\left\Vert\varepsilon\right\Vert^2-\left\Vert\varepsilon^h\right\Vert^2$	$\dfrac{q_0{}^2L^3}{12EA}$	$\dfrac{P^2L(\ln 3-1)}{2EA}$

EXAMPLE 1. ***Cantilever bar analysis with a single linear bar element.***

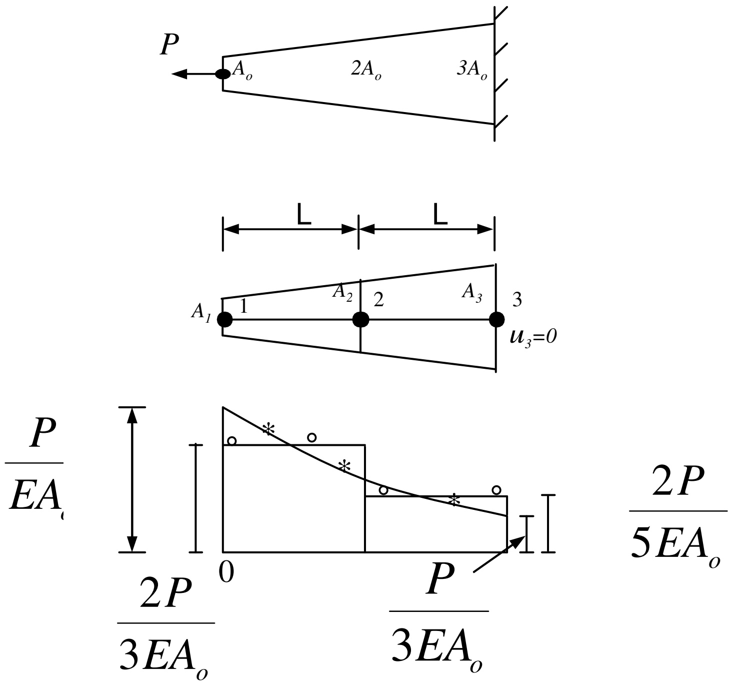




SPURIOUS OSCILLATIONS OF FORCE RESULTANT

EXAMPLE 2. *Cantilever bar analysis with a two linear bar elements.*

Results of the element 1-2



Cantilever bar analysis for the linearly tapering bar with two linear bar elements.

Rigidity matrix [D]	$\frac{EA_0}{2}(3+\xi)$
Orthogonal basis vector spanning 1-D B subspace	$\{v\}_1 = 1$
Analytical strain vector $\{\varepsilon\}$	$\frac{2P}{EA_0(3+\xi)}$
$\{\varepsilon^h\} = \{\bar{\varepsilon}\} = \frac{<\varepsilon, v_1>}{<v_1, v_1>}\{v_1\}$	$\frac{2P}{3EA_0}$
$\left\Vert \varepsilon - \varepsilon^h \right\Vert^2 = \left\Vert \varepsilon \right\Vert^2 - \left\Vert \varepsilon^h \right\Vert^2$	$\frac{P^2L}{EA_0}\left\{\ln 2 - \frac{2}{3}\right\}$

3.2.2 Case B: When the Analytical Reactions are NOT conserved by FEA

$$\{R^{he}\} \neq \{R^{e}\}$$

$$\begin{aligned} a(u^h,u^h)^e - a(u^h,u)^e &= \lfloor u^h, R^{he} - R^e \rfloor_B \\ &= \{\delta^e\}^T \{R^{he} - R^e\} \\ &\neq 0 \end{aligned}$$

- **When the FE computed reactions *{R^h,e^}* disagree with the analytical reactions *{R^e^}*, the approximate (FEA) element strains are best fits to the strains from stiffened analytical solutions u^* with error of the nodal reactions. (With analytical reactions matching the FE reactions).**

$$a(u^h,u^h)^e - a(u^h,u^*)^e = 0$$

EXAMPLE 3.

Fixed-fixed tapered bar analysis with two linear bar elements.

Case : Uniform loading q = 1

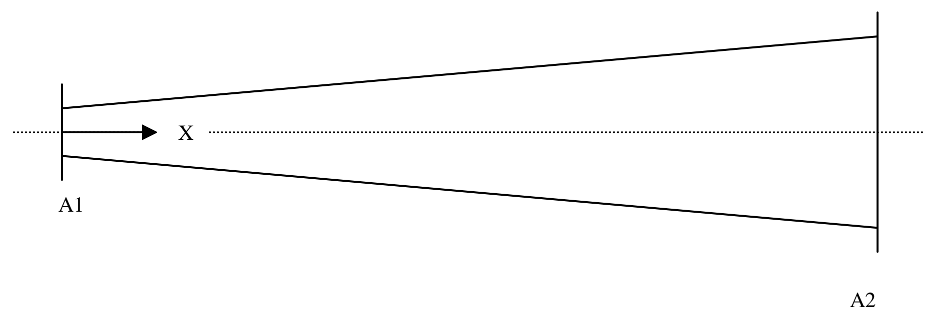
$E = 1$, $l = 1$, $A_2 = 1$ and $A_1 = 0.01$, $u(x=0)=u(x=l)=0$

The exact (analytical) solutions for this case :

Displacement : $u = -1.010101x + 0.219341 \ln(1 + 99x)$

Strain: $\varepsilon = -1.010101 + 21.714759 / (1 + 99x)$

Stress resultant: $Q = 0.2070 - x$



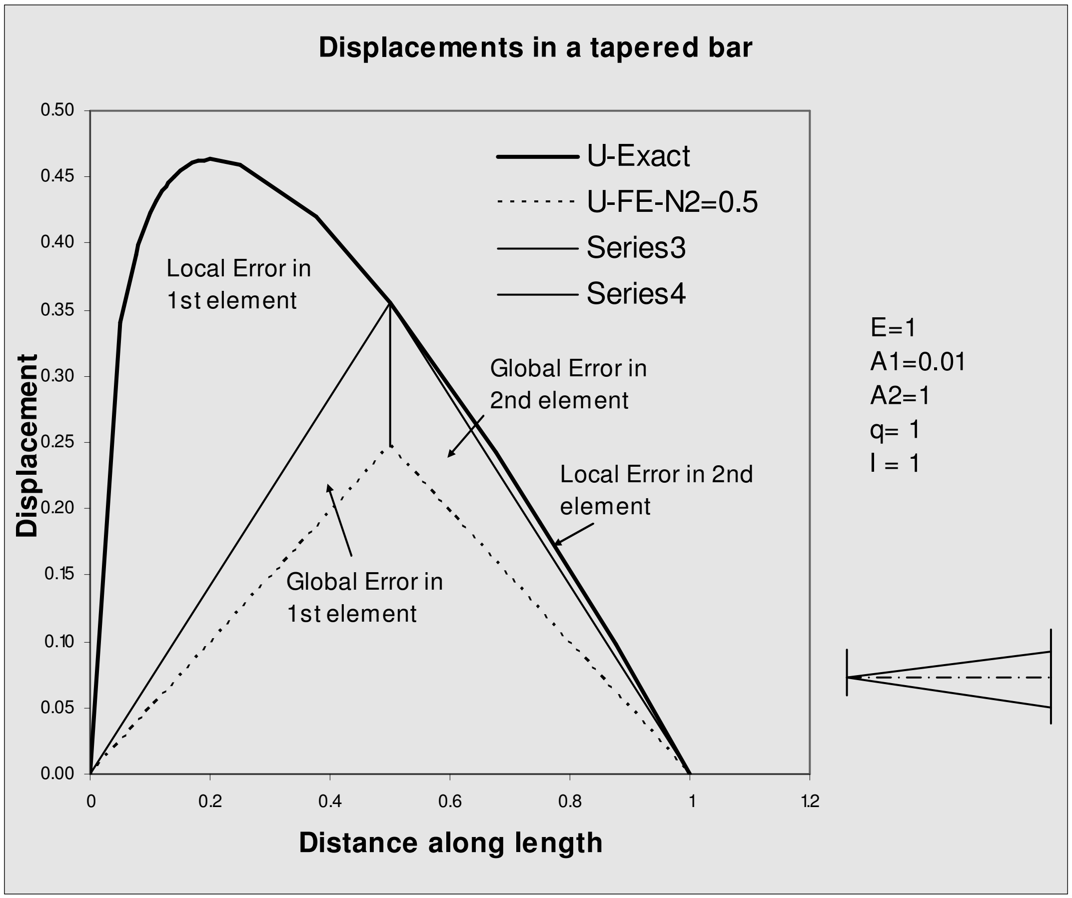
EXAMPLE 3. *Fixed-fixed tapered bar analysis with two linear bar elements.*

Case: Uniform loading $q = 1$

Element e	*1*	*2*
FEM Strain $\{\varepsilon^{he}\}$	*0.4950*	*-0.4950*
Best-fit Strain $\{\overline{\varepsilon}^{e}\}$	*-0.1671*	*-0.7216*
$\{\varepsilon^{he}\}-\{\overline{\varepsilon}^{e}\}$	*0.6621*	*0.2266*
FEM Reaction $\{R^{he}\}$	*-0.3775* *-0.1225*	*0.1225* *-0.6225*
Analytical Reaction $\{R^{e}\}$	*-0.2070* *-0.2930*	*0.2930* *-0.7930*
Reaction Error $\{R^{he}\}-\{R^{e}\}$	*-0.1705* *0.1705*	*-0.1705* *0.1705*

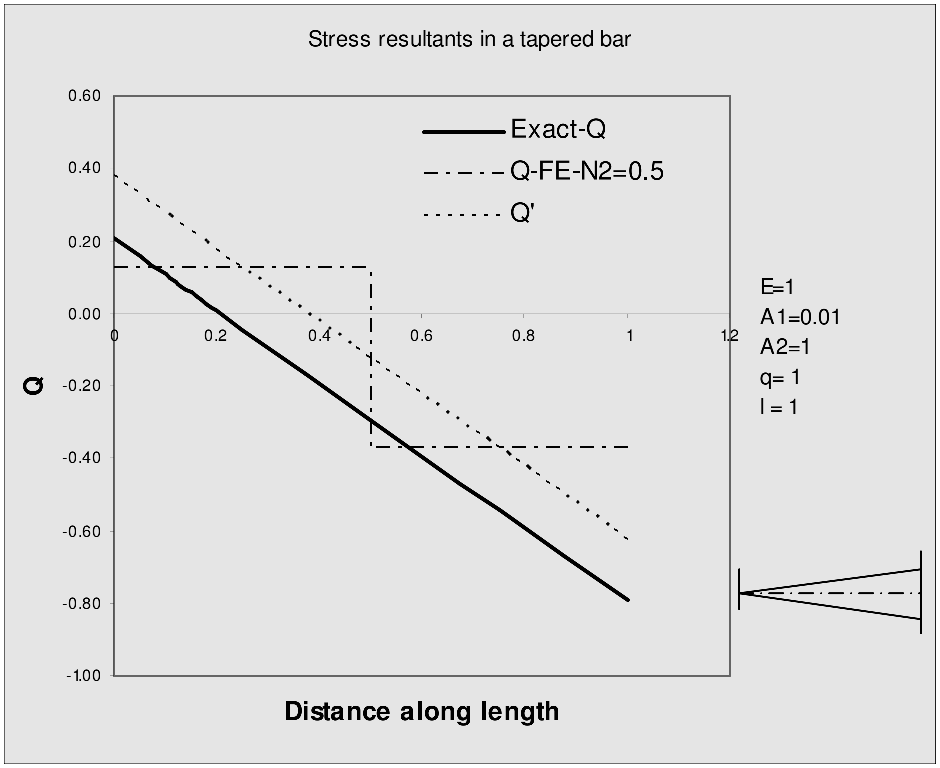
EXAMPLE 3. ***Fixed-fixed tapered bar analysis with two linear bar elements.***

Case: Uniform loading $q = 1$



EXAMPLE 3. *Fixed-fixed tapered bar analysis with two linear bar elements.*

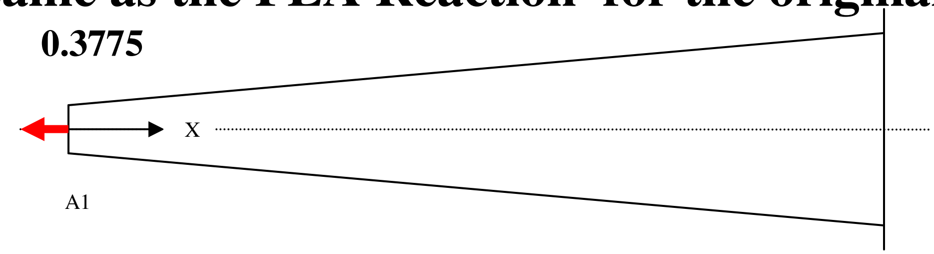
Case : Uniform loading q = 1



The FEA solution suffers an additional spurious stiffening from a tension of 0.1705 (force units) which originates from the error in the nodal reaction vectors. FEA strains are best fits to the stiffened analytical strains.

Modified problem to EXAMPLE 3. ***Free-fixed tapered bar analysis with two linear bar elements.***

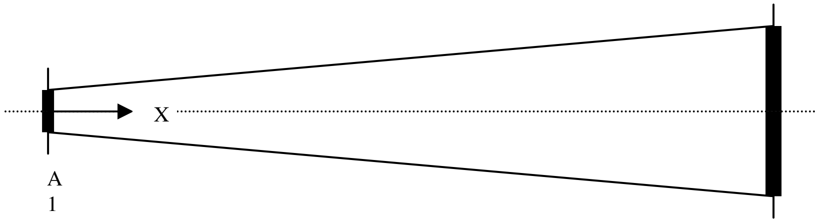
Case : Uniform loading q = 1 and point load P=0.3775 at the free end same as the FEA Reaction for the original fixed end of example 3.



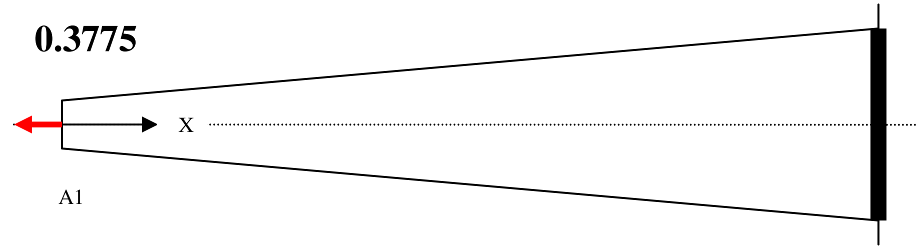
Element e	*1*	*2*
FEM Strain $\{\varepsilon^{he}\}$	*0.4950*	*-0.4950*
Best-fit Strain $\{\bar{\varepsilon}^{e}\}$	*0.4950*	*-0.4950*
$\{\varepsilon^{he}\}-\{\bar{\varepsilon}^{e}\}$	*0*	*0*
FEM Reaction $\{R^{he}\}$	*-0.3775* *-0.1225*	*0.1225* *-0.6225*
Analytical Reaction $\{R^{e}\}$	*-0.3775* *-0.1225*	*0.1225* *-0.6225*
Reaction Error $\{R^{he}\}-\{R^{e}\}$	*0* *0*	*0* *0*

The FEA strain of this problem exactly agrees with its own best-fit (Reactions agree).

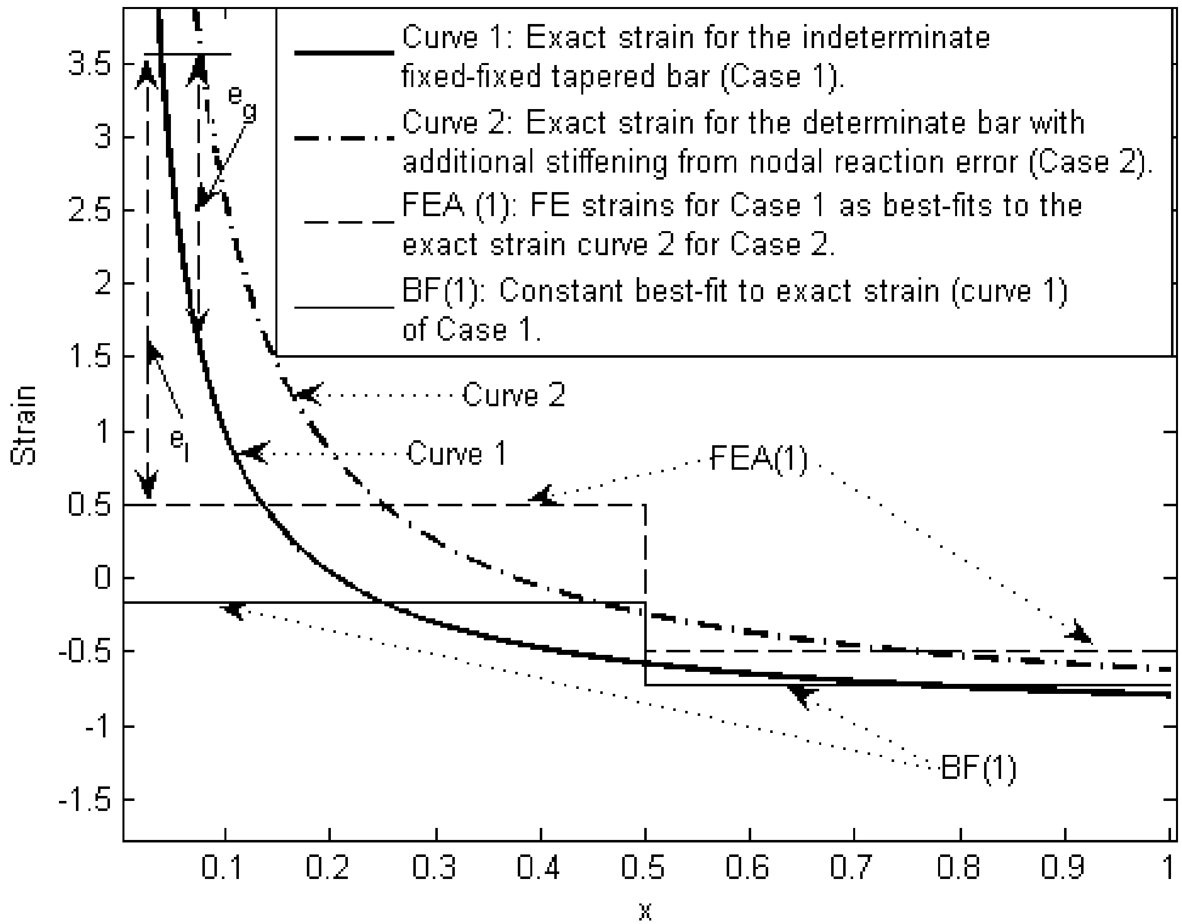
FEA strain of Example 3 (FIXED-FIXED) responds as the best-fit strain of this modified problem, (with additional stiffening of 0.1705 units of force from reaction error that stiffens the original system of Example 3).



Case 1. Fixed-fixed indeterminate bar with u.d.l. q=1



Case 2. Free-fixed bar with u.d.l. q=1, and Load P=0.3775 (⟵) at the free left end



Example 4: *The spherically symmetric Laplace Equation*

The Laplace Equation**:** $$\nabla^2 u = 0$$

The spherically symmetric Laplace Equation: $$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$$

Analytical Solution for Potential u (disp) and Potential Gradient (strain) : $$u(r) = \frac{1}{r} \qquad \varepsilon = \frac{du}{dr} = -\frac{1}{r^2}$$

Satisfies Boundary Conditions (DD): $$u(r_1) = u_1 = \frac{1}{r_1}; \qquad u(r_{n+1}) = u_{n+1} = \frac{1}{r_{n+1}}$$

Boundary Conditions (DD) are applied to the ends of the continuum discretised with *n* elements:

The Weak form in an element '*i*': $$a\left(u,u^h\right) = \left(R_i u_i + R_{i+1}u_{i+1}\right)$$

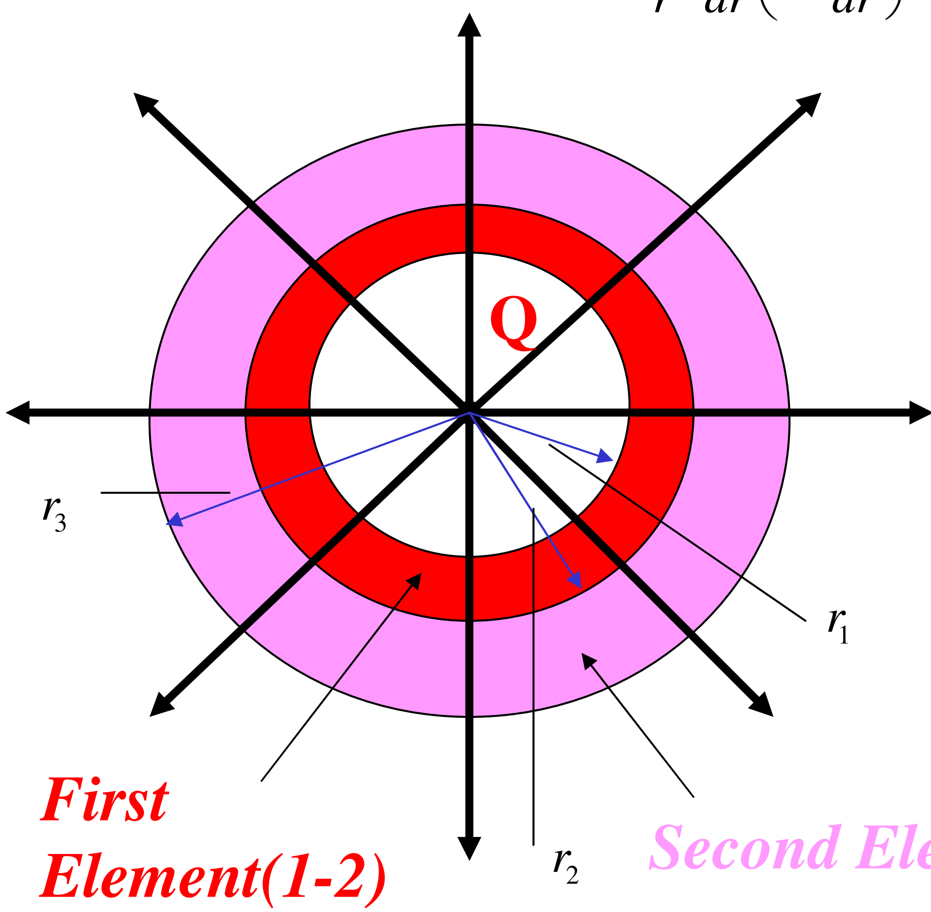
The Bilinear form: $$a(u,u^h) = \int_{r_i}^{r_{i+1}} r^2 \frac{du}{dr}\frac{du^h}{dr}dr$$

Analytical Reactions (Flux): $$R_i = \left(-r^2\frac{du}{dr}\right)_{r=r_i} \qquad R_{i+1} = \left(r^2\frac{du}{dr}\right)_{r=r_{i+1}}$$

ELECTRIC FIELD FROM A POINT CHARGE Q

An Example of Application of Spherically Symmetric Laplace Equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)=0$$



Potential : $u=\dfrac{Q}{4\pi\varepsilon_0 r}$

Intensity: $E=-\dfrac{du}{dr}=\dfrac{Q}{4\pi\varepsilon_0 r^2}$

Here $\dfrac{Q}{4\pi\varepsilon_0}=k_2=1$

The Continuum is discretised into 2 Linear Finite Elements.

FE Solution of Symmetric Laplace Equation (using 2 linear elements)

Original Problem with (DD)

FE Flux=2.1645 **FE Flux=-2.1645**

Anal Flux=1 (Incoming at 1)

Anal Flux= -1 (Outgoing at 3)

$$r_1 = 0.1, \quad r_2 = 0.2, \quad r_3 = 10$$

$$u_1 = 1/r_1 = 10, \quad u_3 = 1/r_3 = 0.1$$

Element e	1	2
FEM Strain $\{\varepsilon^{he}\}$	-92.764	-0.0636
Best-fit Strain $\{\overline{\varepsilon}^e\}$	-42.8571	-0.0294
$\{\varepsilon^{he}\}-\{\overline{\varepsilon}^e\}$	-49.9065	-0.0342
FEM Reaction $\{R^{he}\}$	2.1645 -2.1645	2.1645 -2.1645
Analytical Reaction $\{R^e\}$	1.0 -1.0	1.0 -1.0
Reaction Error $\{R^{he}\}-\{R^e\}$	1.1645 -1.1645	1.1645 -1.1645

Modified problem with (DN)

Applied End Force $F_3 = -2.1645$

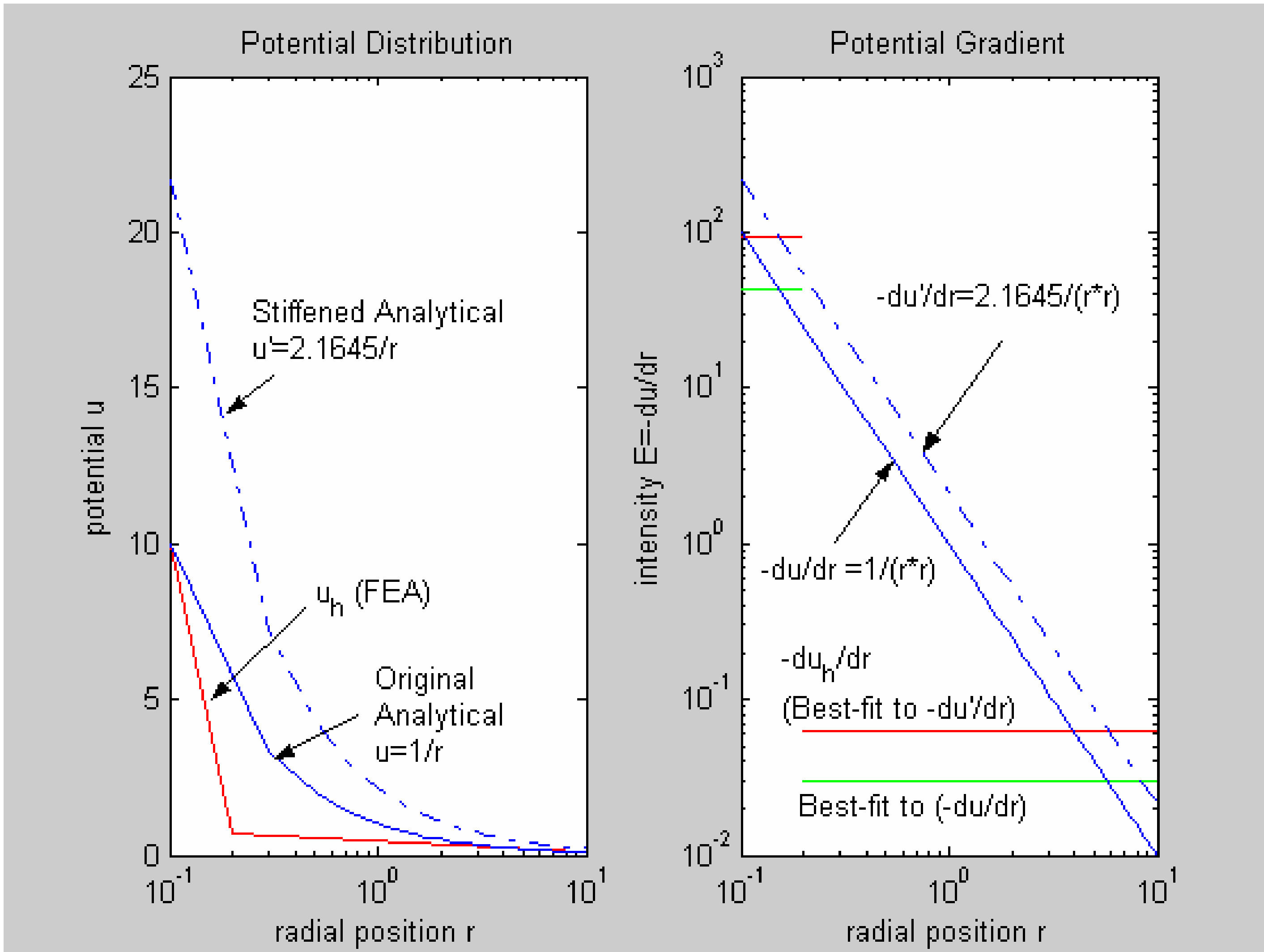
Flux=2.1645 (Incoming at 1)

Flux= -2.1645 (Outgoing at 3)

$$u_1 = 1/r_1 = 10, \quad F_3 = -2.1645$$

Element e	1	2
FEM Strain $\{\varepsilon^{he}*\}$	-92.764	-0.0636
Best-fit Strain $\{\overline{\varepsilon}^e*\}$	-92.764	-0.0636
$\{\varepsilon^{he}*\}-\{\overline{\varepsilon}^e*\}$	0.0	0.0
FEM Reaction $\{R^{he}\}$	2.1645 -2.1645	2.1645 -2.1645
Analytical Reaction $\{R^e\}$	2.1645 -2.1645	2.1645 -2.1645
Reaction Error $\{R^{he}\}-\{R^e\}$	0.0 0.0	0.0 0.0

FE Solution of Symmetric Laplace Equation (using 2 linear elements)



Bilinear symmetric forms and the corresponding strains for different elements

Bar element	$a(u^h,u)^e = \int_e \left\{\frac{du^h}{dx}\right\}^T EA\left\{\frac{du}{dx}\right\}dx$ $a(u^h,u)^e = \int_e \{\varepsilon^h\}^T EA\{\varepsilon\}dx = <\varepsilon^h,\varepsilon>$	$\{\varepsilon^h\} = \frac{du^h}{dx}$
Euler beam element	$a(w^h,w)^e = \int_e \left\{\frac{d^2w^h}{dx^2}\right\}^T EI\left\{\frac{d^2w}{dx^2}\right\}dx$ $a(w^h,w)^e = \int_e \{\varepsilon^h\}^T EI\{\varepsilon\}dx = <\varepsilon^h,\varepsilon>$	$\{\varepsilon^h\} = \frac{d^2w^h}{dx^2}$
Timoshenko beam element	$a(u^h,u)^e = \int_e \{\varepsilon^h\}^T \begin{bmatrix} EI & 0 \\ 0 & kGA \end{bmatrix}\{\varepsilon\}dx$ $= <\varepsilon^h,\varepsilon>$	$\{\varepsilon^h\} = \begin{Bmatrix} d\theta^h / dx \\ (\theta^h - dw^h / dx \end{Bmatrix}$
Euler beam element on elastic foundation with stiffness *k* per *unit length*	$a(w^h,w)^e = \int_e \{\varepsilon^h\}^T \begin{bmatrix} EI & 0 \\ 0 & k \end{bmatrix}\{\varepsilon\}dx$ $= <\varepsilon^h,\varepsilon>$	$\{\varepsilon^h\} = \begin{Bmatrix} \frac{d^2w^h}{dx^2} \\ w^h \end{Bmatrix}$

3.3 The Simple Euler Beam Element (Cubic)

The differential equation for the Euler beam with section rigidity *EI* and distributed loading *q(x)* is given by

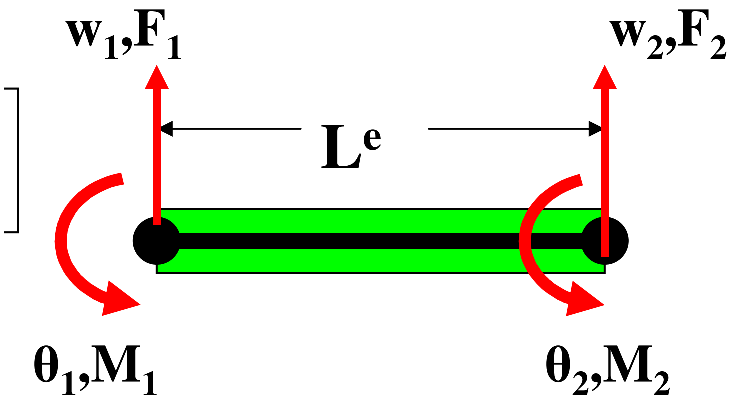
$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)-q=0 \qquad (3.11)$$

Virtual work principle leads to the Galerkin Equation in the element 'e'.

$$\boxed{a(w^h,w)^e=(w^h,q)^e+\left\{\delta^e\right\}^T\left\{R^e\right\}} \qquad (3.12)$$

$$\int_e \frac{d^2w^h}{dx^2}\left(EI\frac{d^2w}{dx^2}\right)dx=\int_e w^h q\,dx+\left[w_1{}^h R_1{}^e{}_{(x=x_1)}+\left(\frac{dw^h}{dx}\right)_{x=x_1}R_2{}^e{}_{(x=x_1)}\right]$$

$$+\left[w_2{}^h R_3{}^e{}_{(x=x_2)}+\left(\frac{dw^h}{dx}\right)_{x=x_2}R_4{}^e{}_{(x=x_2)}\right]$$



The Analytical Reactions are:

$$R_1^e=\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)_{x=x_1} \qquad R_2^e=-\left(EI\frac{d^2w}{dx^2}\right)_{x=x_1}$$

$$R_3^e=-\frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right)_{x=x_2} \qquad R_4^e=\left(EI\frac{d^2w}{dx^2}\right)_{x=x_2} \qquad (3.13)$$

The weak form for FEA $a(w^h, w^h)^e = (w^h, q)^e + \{\delta^e\}\{R^{he}\}$

$$\int_e \frac{d^2 w^h}{dx^2}\left(EI\frac{d^2 w^h}{dx^2}\right)dx = \int_e w^h q dx + \left[w_1^{\,h} R_1^{\,he}{}_{(x=x_1)} + \left(\frac{dw^h}{dx}\right)_{x=x_1} R_2^{\,he}{}_{(x=x_1)} \right] + \left[w_2^{\,h} R_3^{\,he}{}_{(x=x_2)} + \left(\frac{dw^h}{dx}\right)_{x=x_2} R_4^{\,he}{}_{(x=x_2)} \right] \tag{3.14}$$

The FEA computed nodal reactions can be expressed as

$$\{R^{h,e}\} = [R_1^{\,e} \quad R_2^{\,e} \quad R_3^{\,e} \quad R_4^{\,e}]^T = [K^e]\{\delta^e\} - \{F^e\} \tag{3.15}$$

where

$$[K^e] = \int_e [B]^T EI[B]dx \qquad \{F^e\} = \int_e [N]^T q(x)dx$$

3.3.1 The B Subspace and Strain Projection for the Simple Euler Beam Element (Cubic)

Element formulation with Hermite Cubics

$$Disp: \quad w^h = [N_1 \quad N_2 \quad N_3 \quad N_4] \ \{\delta^e\}, \qquad \{\delta^e\} = \begin{bmatrix} w_1 & \left(\dfrac{dw^h}{dx}\right)_1 & w_2 & \left(\dfrac{dw^h}{dx}\right)_2 \end{bmatrix}^T$$

$$Strain: \varepsilon^h = d^2 w^h / dx^2 = [B]\{\delta^e\}$$

$$Element\ Rigidity \quad D = EI, \quad Inner\ Product \quad <a,b> = \int_{-1}^{1} \{a\}^T EI \{b\}.\frac{L^e}{2} d\xi$$

The dimension of the subspace B is also the rank of the stiffness matrix, *m=dim (B)=N-R.*

Here N= Number of total degrees of freedom of element=4

R=Number of rigid body motions of element=2

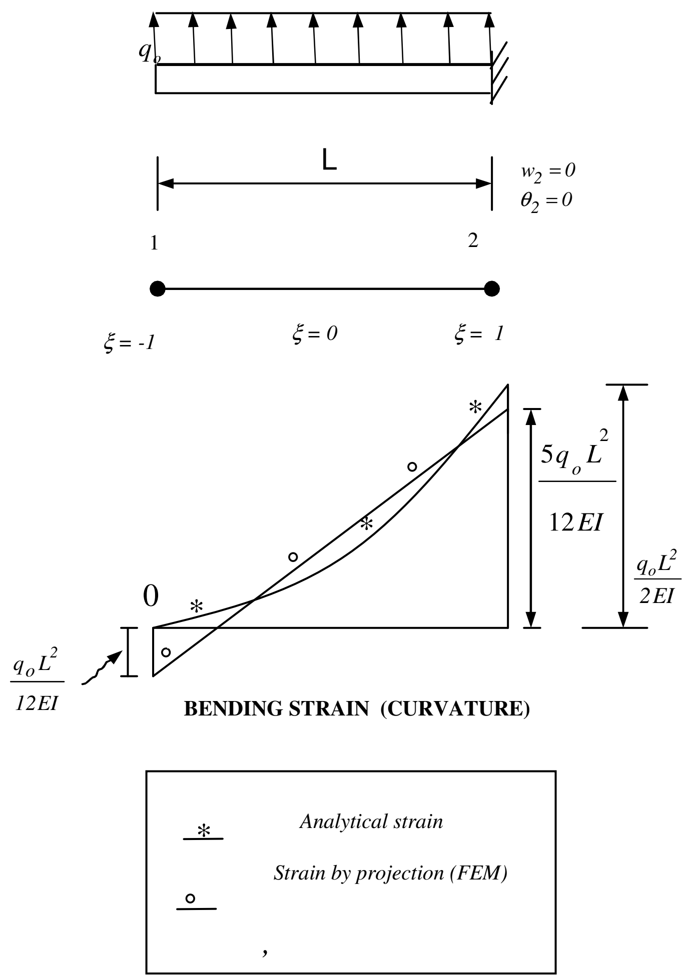
(m=4-2=2 for the Euler beam element).

Projection rule: $$\{\varepsilon^h\} = \{\bar{\varepsilon}\} = \sum_{j=1}^{m} \frac{<\varepsilon, v_j>}{<v_j, v_j>} \{v_j\}, \qquad <v_i, v_j> = 0 \quad for \quad i \neq j$$

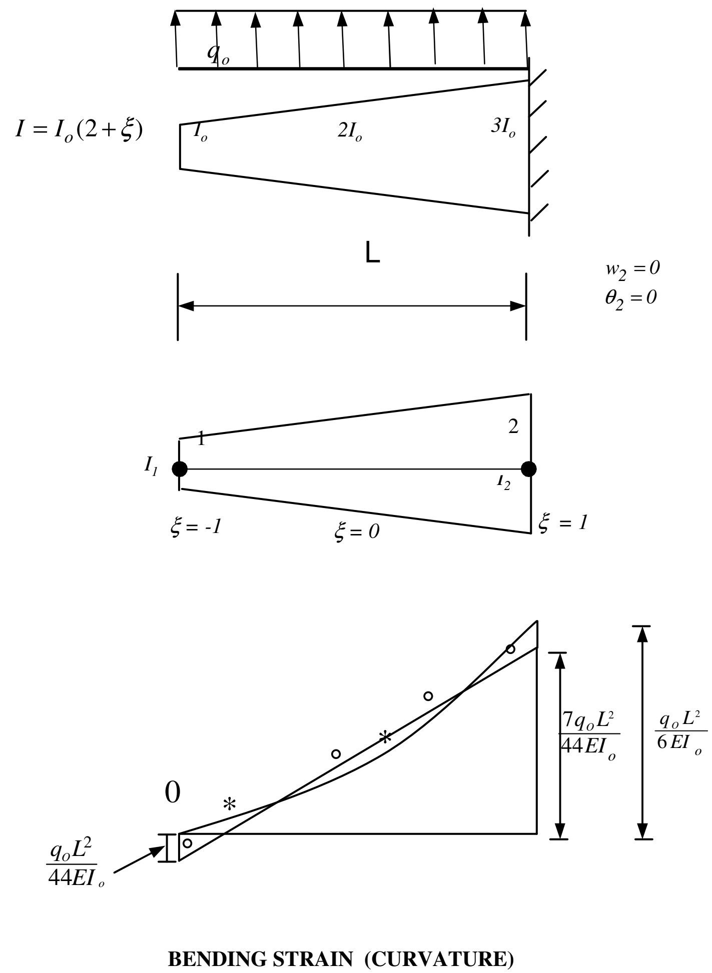
Here ε is the analytical strain (a stiffened one from nodal reaction error, if any).

Examples: Cantilever beam analysis with single Euler element

(a)



(b



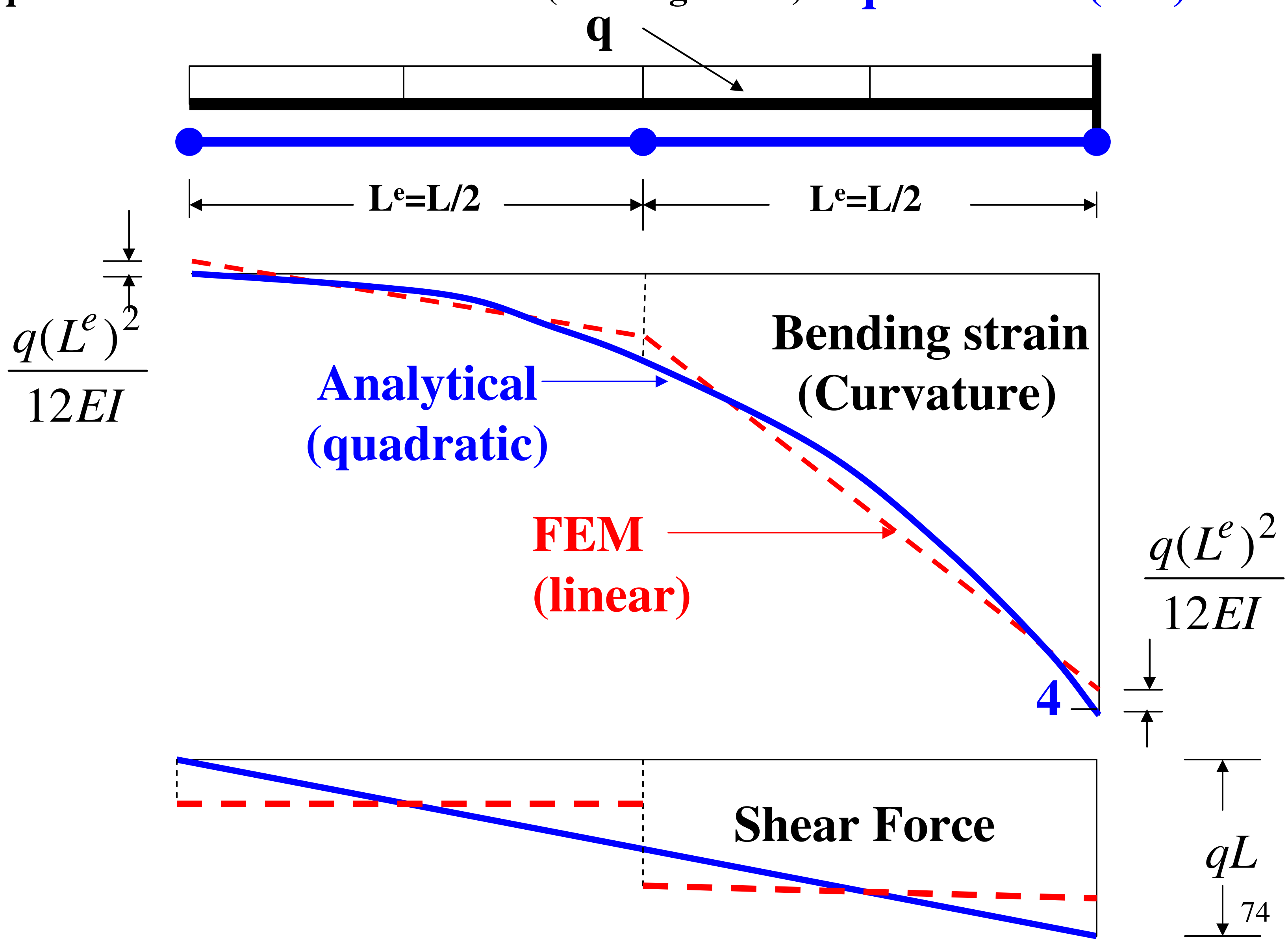
Examples: Cantilever beam analysis (..Continued)

	(a) Constant sectional moment of inertia.	**(b)** Linearly varying sectional moment of inertia. ($I_1=I_o$, $I_2=3I_o$)
Rigidity matrix $[D(\xi)]$	EI	$EI_o(2+\xi)$
Analytical strain vector $\{\varepsilon\}$	$\frac{q_o L^2 (1+\xi)^2}{8EI}$	$\frac{q_o L^2 (1+\xi)^2}{8EI_o(2+\xi)}$
Orthogonal basis vectors,	$\{u_1\}=\{1\}$, $\{u_2\}=\{\xi\}$	$\{u_1\}=\{1\}$, $\{u_2\}=\{(6\xi-1)\}$
FEM strain = strain by projection $\{\varepsilon^h\}=\{\bar{\varepsilon}\}=\sum_{j=1}^{m=2}\frac{<\varepsilon,u_j>}{<u_j,u_j>}\{u_j\}$	$\frac{q_o L^2 (2+3\xi)}{12EI}$	$\frac{q_o L^2 (3+4\xi)}{44EI_o}$
The energy error rule $\left\|\varepsilon-\varepsilon^h\right\|^2=\left\|\varepsilon\right\|^2-\left\|\varepsilon^h\right\|^2=$	$\frac{q_o^2 L^5}{720EI}$	$\frac{q_o^2 L^5}{128EI_o}\left\{ln(3)-\frac{12}{11}\right\}$

CANTILEVER BEAM ANALYSIS USING TWO EULER BEAM ELEMENTS

Uniformly distributed loading is q per unit length.

q is such that fixed end curvature (bending strain) is $qL^2/2EI=4\ (m^{-1})$



Lecture 3

Chapter 4

Consequence of the best-fit nature in FEA.
Optimal points (Gauss's and Prathap's points) for exact strain recovery

4.1 The Gauss points as the optimal points

(For statically determinate problems)

Review of cases where the analytical strains that are polynomials just one order higher than the polynomial representing the approximate strains.

Exact (analytical) $$\varepsilon = \sum_{i=1}^{N+1} a_i P_i(\xi) \qquad -1 \le \xi \le 1$$

Approximate as best-fits $$\varepsilon^h = \sum_{i=1}^{N} a_i P_i(\xi)$$

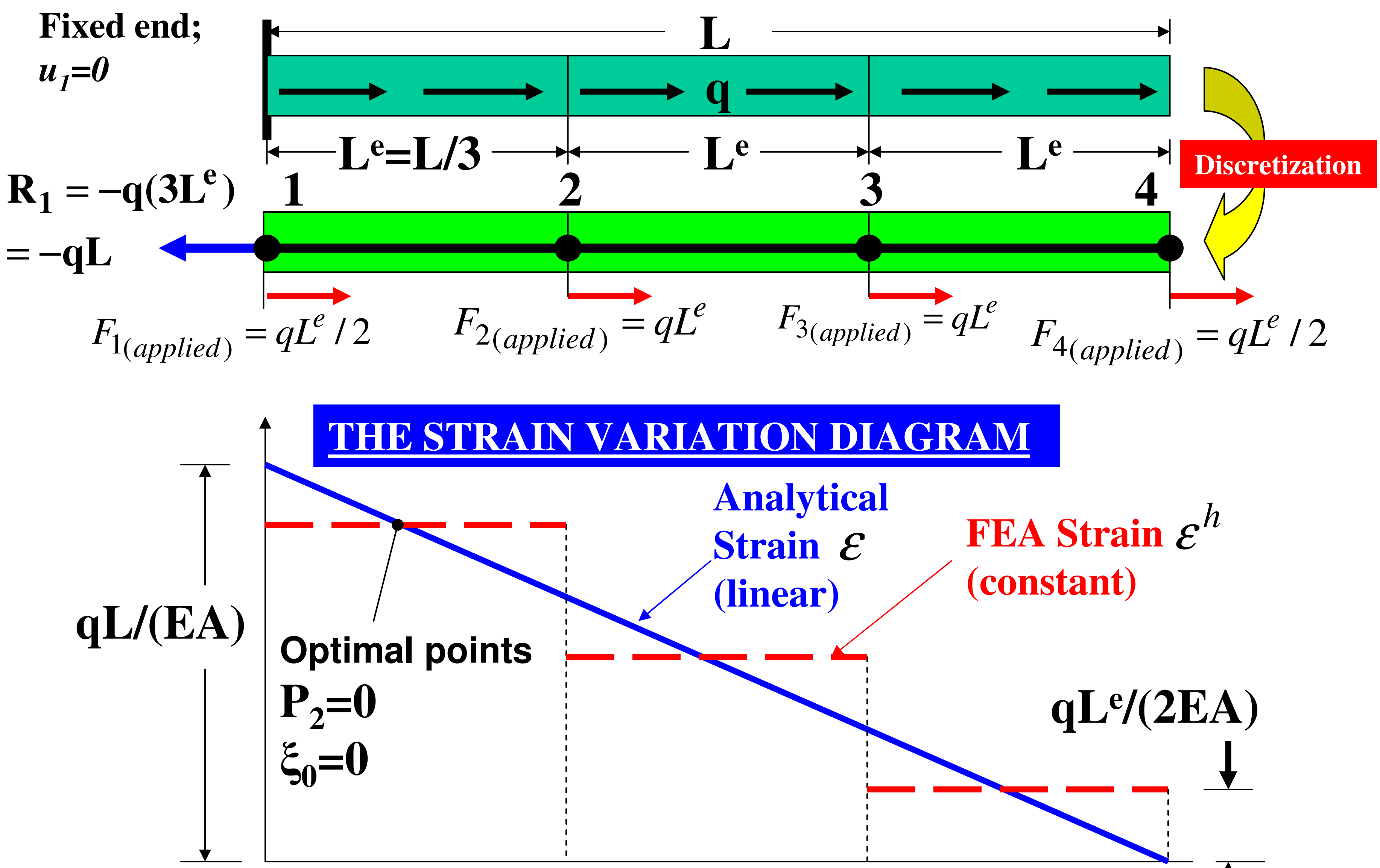
At the optimal points, the exact strains are captured.

$$\varepsilon^h = \varepsilon$$

Thus the optimal points are determined from the roots of the highest Legendre Polynomial $P_{N+1}(\xi)$. These are called the Gauss points.

$$P_{N+1}(\xi) = 0$$
$$\Rightarrow \xi_0 = \xi_{Gauss}$$

Example 1. Analysis of a bar under u.d.l. q using FEM



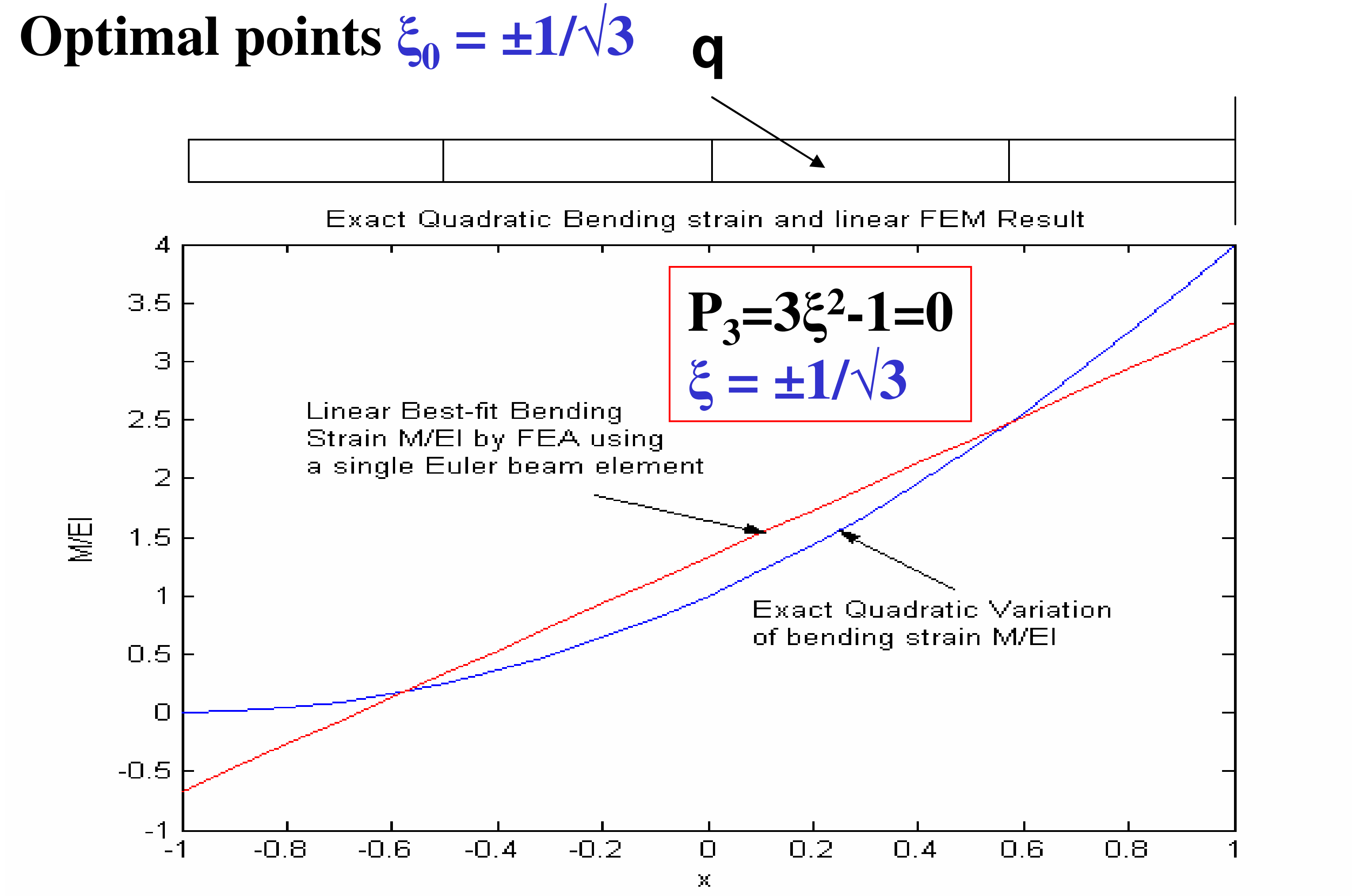
FEA gives the best-fit to the analytical strain. Why ?

Example 2. Cantilever beam analysis using a single euler beam element of length L

Uniformly distributed loading is q per unit length.

q is such that fixed end curvature (bending strain) is $qL^2/2EI=4$ (m^{-1})

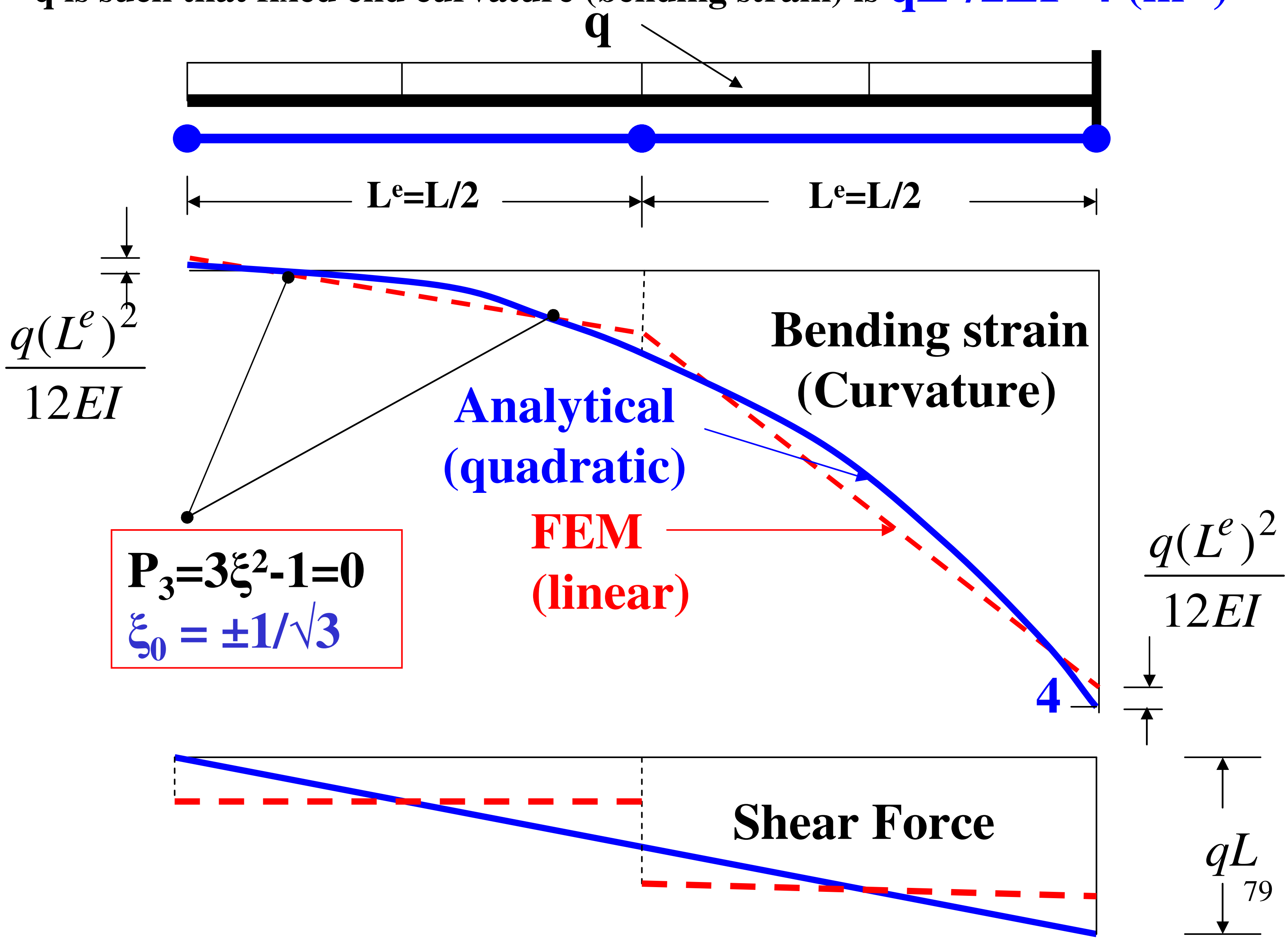
Optimal points $\xi_0 = \pm 1/\sqrt{3}$



Example 3. Cantilever beam analysis using two euler beam elements

Uniformly distributed loading is q per unit length.

q is such that fixed end curvature (bending strain) is $qL^2/2EI=4$ (m^{-1})



4.2 Prathap points as general optimal points (for statically determinate problems)

For cases where the polynomial of the analytical strain is more than one order higher than the polynomial for the approximate strain.

Exact (analytical) $\varepsilon = \sum_{i=1}^{N+r} a_i P_i(\xi) \qquad -1 \le \xi \le 1$

Approximate as best-fits $\varepsilon^h = \sum_{i=1}^{N} a_i P_i(\xi)$

At the optimal points, the exact strains are captured.

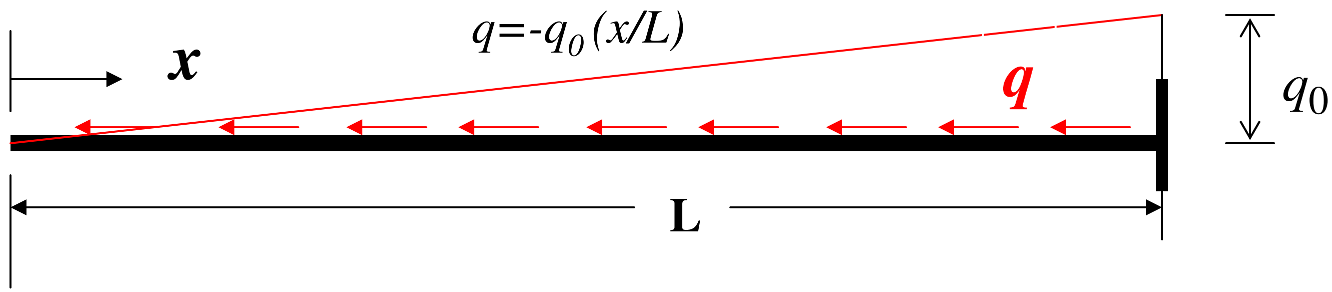
$$\varepsilon^h = \varepsilon$$

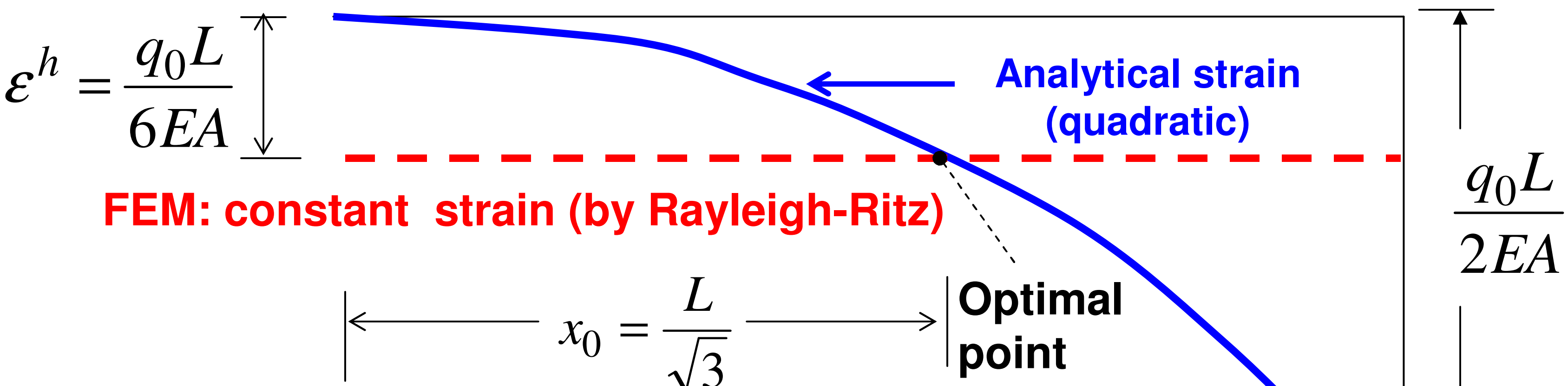
Thus the optimal points are determined from the roots of the following expressions. These are the Prathap points.

$$\boxed{\begin{array}{l} a_{N+1}P_{N+1}(\xi) + a_{N+2}P_{N+2}(\xi) \ldots\ldots + a_{N+r}P_{N+r}(\xi) = 0 \\ \Rightarrow \xi_0 = \xi_{Prathap} \end{array}}$$

Gauss points are thus special cases of Prathap points

Example 4. Cantilever bar analysis using one linear element under linearly varying axial load distribution





$$\text{Optimal Point} \qquad \varepsilon^h = \varepsilon \quad \Rightarrow x_0 = \frac{L}{\sqrt{3}} \quad \xi_0 = \frac{2x_0}{L} - 1 = -\left(1 - \frac{2}{\sqrt{3}}\right)$$

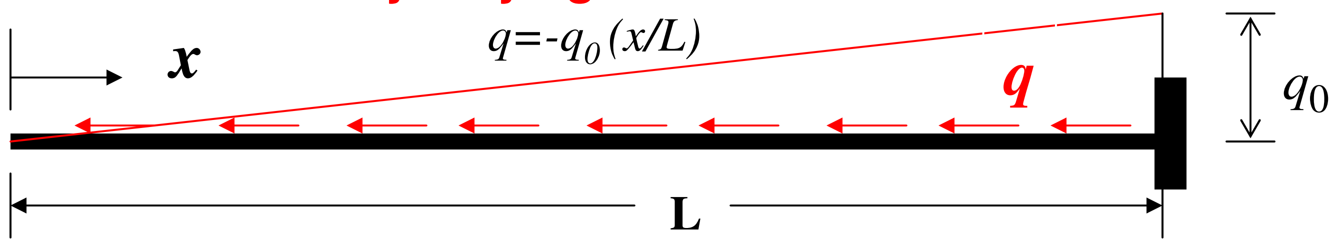
Analytical:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = -q = -(-q_0 \frac{x}{L}) \qquad \varepsilon = \frac{q_0}{2EA}\left(\frac{x^2}{L}\right)$$

Rayleigh-Ritz:

$$u^h(x) = -a(L-x) \qquad u(L) = 0 \qquad \varepsilon^h = \frac{du^h}{dx} = \frac{q_0 L}{6EA}$$

Rayleigh –Ritz calculations for Example 4
Cantilever bar analysis using one element
under linearly varying axial load distribution



Using a linear displacement function (or a constant strain)

$$u^h(x) = -a(L-x) \qquad u(L) = 0$$

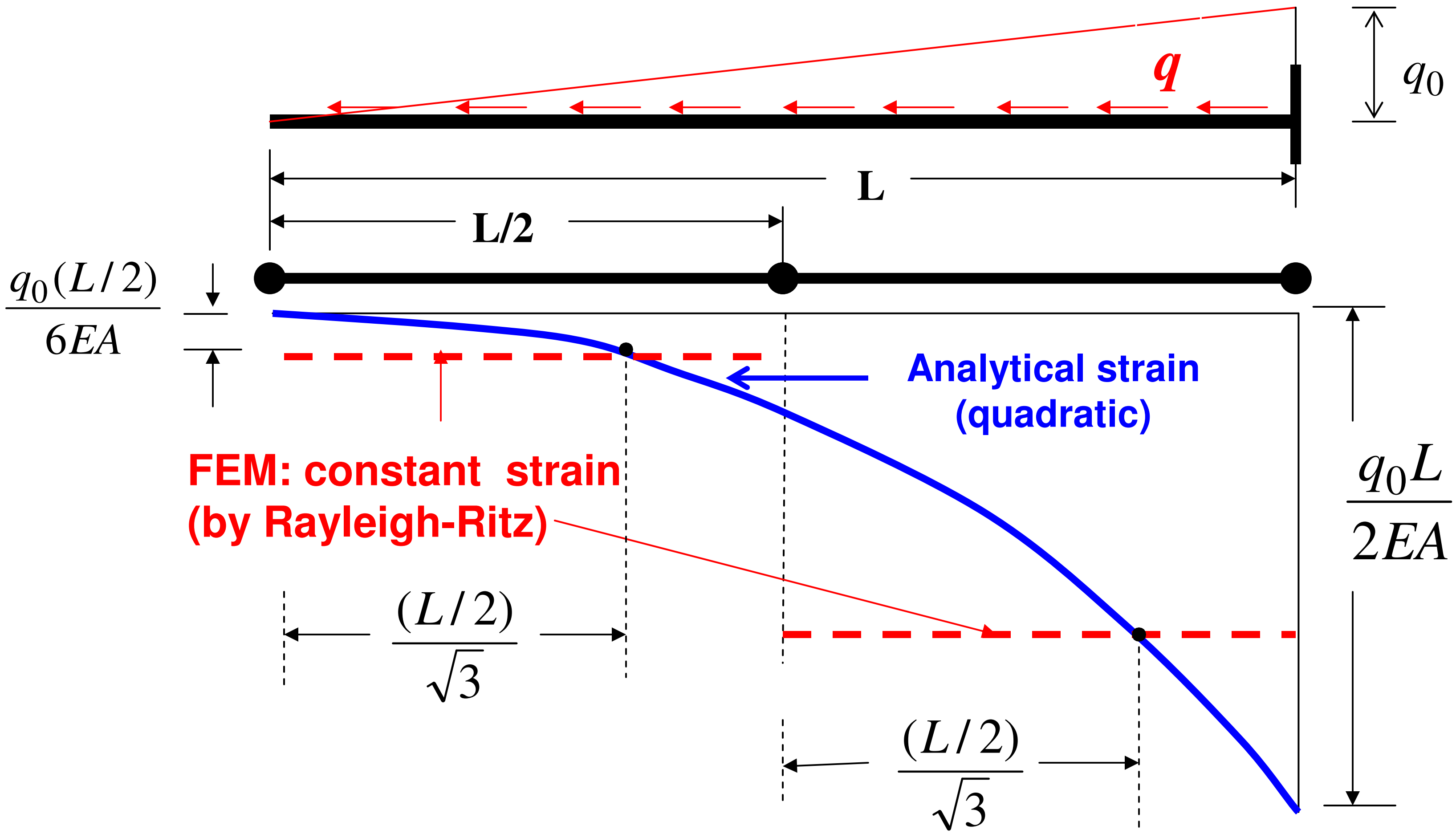
$$\varepsilon^h = \frac{du^h}{dx} = a$$

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du^h}{dx}\right)^2 dx - \int_0^L q_0\left(\frac{x}{L}\right)\left(u^h\right)dx = \frac{1}{2}\int_0^L EAa^2 dx - \int_0^L q_0\left(\frac{x}{L}\right)\{-a(L-x)\}dx$$

$$\frac{\partial\Pi}{\partial a} = 0 \qquad \Rightarrow \qquad a = \frac{q_0 L}{6EA}, \quad \varepsilon^h = \frac{du^h}{dx} = a = \frac{q_0 L}{6EA}$$

Example 5. Cantilever bar analysis using two linear elements under linearly varying axial load distribution

$q = -q_0 (x/L)$



4.3 Prathap points as general optimal points

(for statically indeterminate problems)

For cases where the system suffers from spurious stiffening of the system due to nodal reaction errors.

The FE strain is the orthogonal projection (best-fit) of the stiffened analytical strain (from nodal reaction errors)

$$\{\varepsilon^{h^e}\} = \sum_{j=1}^{m} \frac{<\varepsilon^e{}_{stiff}, v_j>}{<v_j, v_j>}\{v_j\}, \qquad <v_i, v_j> = 0 \quad for \quad i \neq j; \quad span(B) = (v_1, v_2, ... v_m)$$

$$\{\varepsilon^{h^e}\} = \left\{\bar{\varepsilon}^e\right\} + \Delta\left\{\bar{\varepsilon}^e\right\}$$

$$\left\{\bar{\varepsilon}^e\right\} = \sum_{j=1}^{m} \frac{<\varepsilon^e, v_j>}{<v_j, v_j>}\{v_j\}; \qquad \Delta\left\{\bar{\varepsilon}^e\right\} = \sum_{j=1}^{m} \frac{<\Delta\varepsilon^e, v_j>}{<v_j, v_j>}\{v_j\}$$

Best-fit of original analytical strain

Best-fit of additional analytical strain from nodal reaction errors

For indeterminate cases, Prathap points are determined from the best-fit strain of the stiffened analytical strain from reaction errors

Example 6.

Fixed-fixed tapered bar analysis with two linear bar elements.

Case : Uniform loading $q = 1$

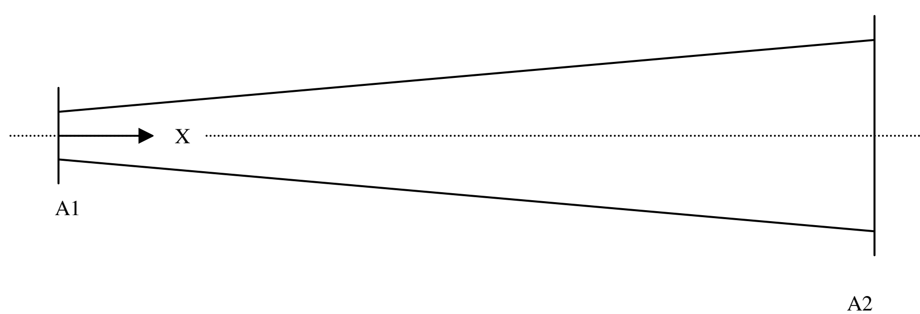
$E = 1$, $l = 1$, $A2 = 1$ and $A1 = 0.01$, $u(x=0)=u(x=l)=0$

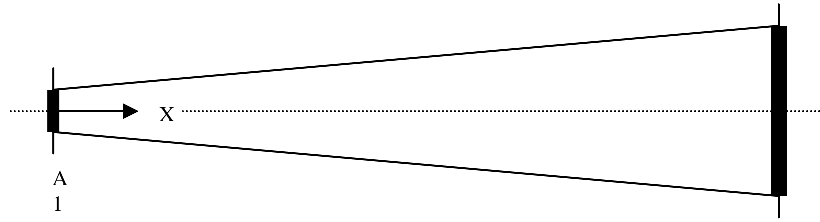
The exact (analytical) solutions for this case :

Displacement : $u = -1.010101x + 0.219341 \ln(1 + 99x)$

Strain: $= -1.010101 + 21.714759 / (1 + 99x)$

Stress resultant: $Q = 0.2070 - x$

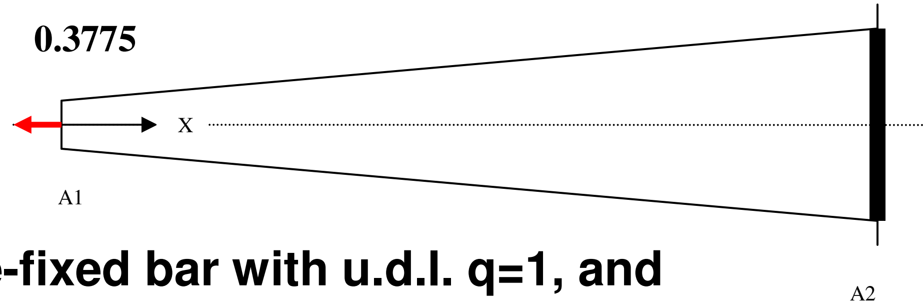




Case 1. Fixed-fixed indeterminate bar with u.d.l. q=1

Node	1	2	3
Position x	0.	0.5	1.0
Sectional Area, A	0.01 0	0.505	1.0
Exact displacement u	0.	0.355 2	0.
FEM displacement u^h	0.	0.247 5	0.
du/dx *Exact strain*	20.7 0	-0.58	- 0.792 9

Element e	1	2
FEM Strain $\{\varepsilon^{he}\}$	0.4950	-0.4950
Best-fit Strain $\{\bar{\varepsilon}^e\}$	-0.1671	-0.7216
$\{\varepsilon^{he}\}-\{\bar{\varepsilon}^e\}$	0.6621	0.2266
FEM Reaction $\{R^{he}\}$	-0.3775 -0.1225	0.1225 -0.6225
Analytical Reaction $\{R^e\}$	-0.2070 -0.2930	0.2930 -0.7930
Reaction Error $\{R^{he}\}-\{R^e\}$	-0.1705 0.1705	-0.1705 0.1705

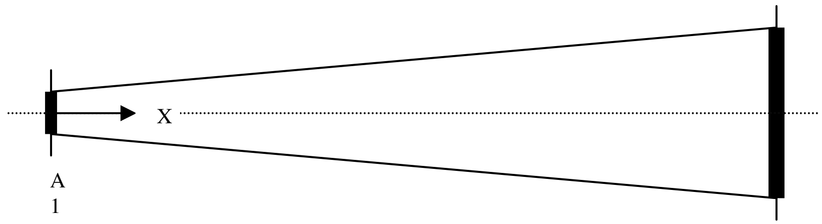


Case 2. Free-fixed bar with u.d.l. q=1, and Load P=0.3775 (←) at the free left end

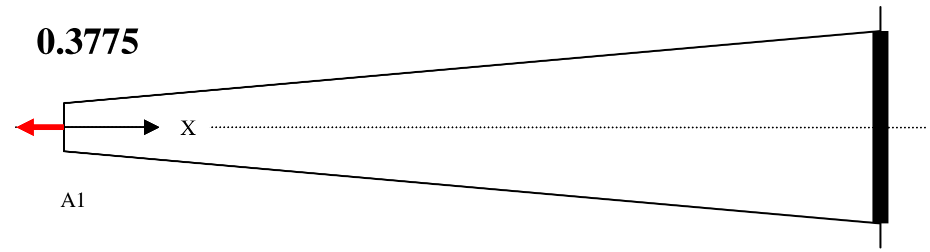
Node	1	2	3
Position x	0.	0.5	1.0
Sectional Area, A	0.010	0.505	1.0
Exact displacement u	-0.7931	0.2376	0.
FEM displacement u^h	0.	0.2475	0.

**The externally applied force at the free end appears as the end reaction.*

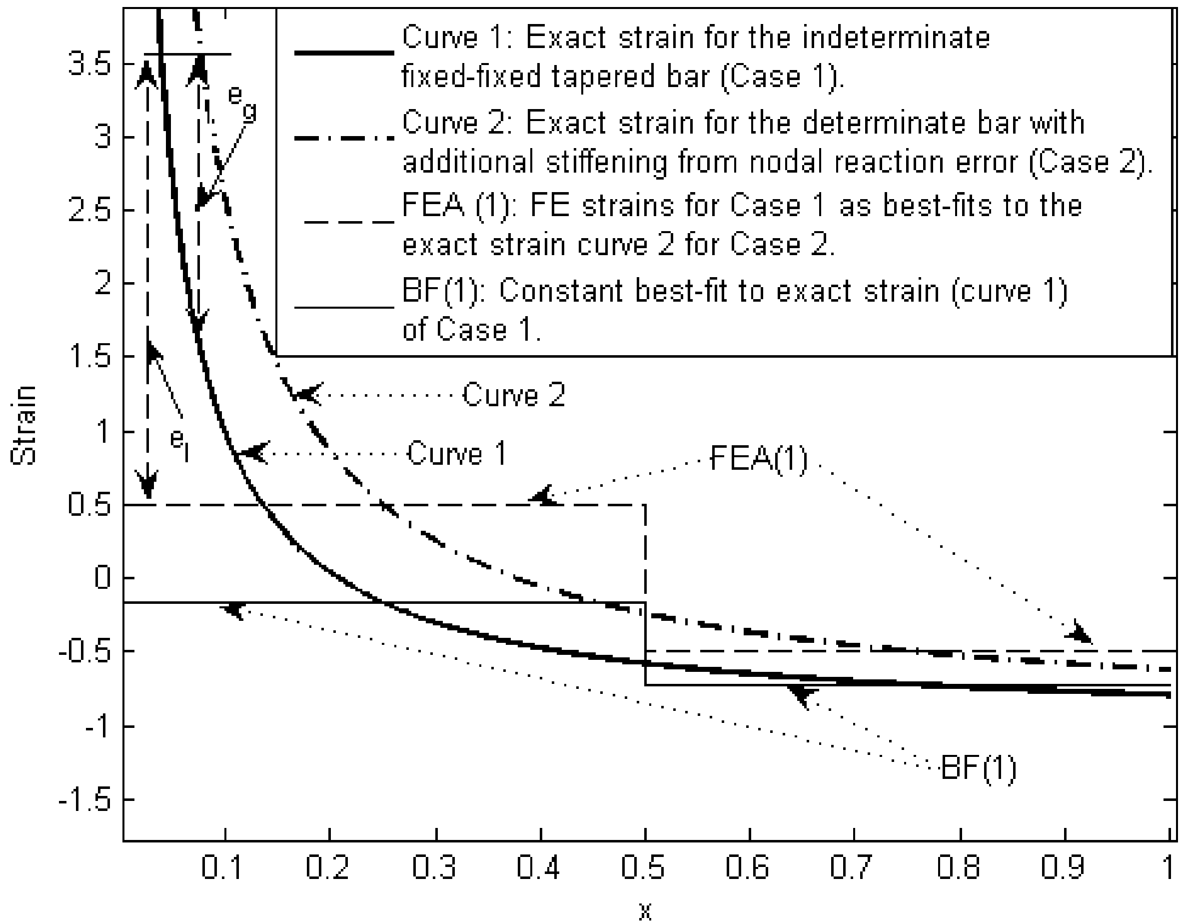
Element e	1	2
$\{\varepsilon^{he}\}$ *FEM Strain*	0.4950	-0.4950
$\{\bar{\varepsilon}^{e}\}$ *Best-fit Strain*	0.4950	-0.4950
$\{\varepsilon^{he}\} - \{\bar{\varepsilon}^{e}\}$	0	0
FEM Reaction $\{R^{he}\}$	-0.3775* -0.1225	0.1225 -0.6225
Analytical Reaction $\{R^{e}\}$	-0.3775* -0.1225	0.1225 -0.6225
Reaction Error $\{R^{he}\} - \{R^{e}\}$	0 0	0 0



Case 1. Fixed-fixed indeterminate bar with u.d.l. q=1



Case 2. Free-fixed bar with u.d.l. q=1, and Load P=0.3775 (←) at the free left end



Summary

A review of linear algebra has been made. The concept of vector spaces and projections are introduced. Orthogonal projections of vectors onto vector spaces (including polynomial spaces) have been demonstrated.

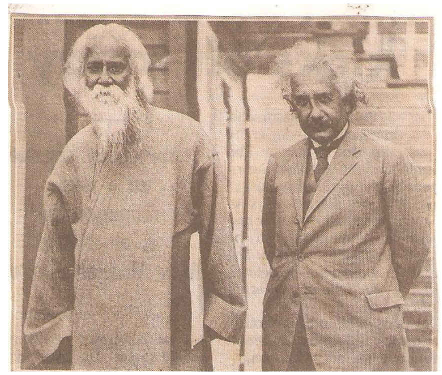
The weak forms that naturally arise in finite elements (from virtual work principles) have been shown in terms of inner products and norms of strain vectors in function spaces. The special cases of one dimensional elements (bar and beam elements) have been clearly demonstrated with suitable examples.

It has been shown how FE formulations actually lead to the 'best-fit paradigm' of strains (and therefore stresses) in the mathematically abstract language of linear algebra. Element strains are best-fits (or orthogonal projections) of analytical strains onto function subspaces B that are generated from the strain-displacement matrices.

For statically determinate systems, the FE strain is the best-fit of the original analytical strain vector. For statically indeterminate systems, displacement (and strain) approximations can lead to errors in nodal reactions, that make the FE strain deviate from the best-fit. However, even for such cases, the FE strain stands as best-fit to the element strain from stiffened analytical solution.

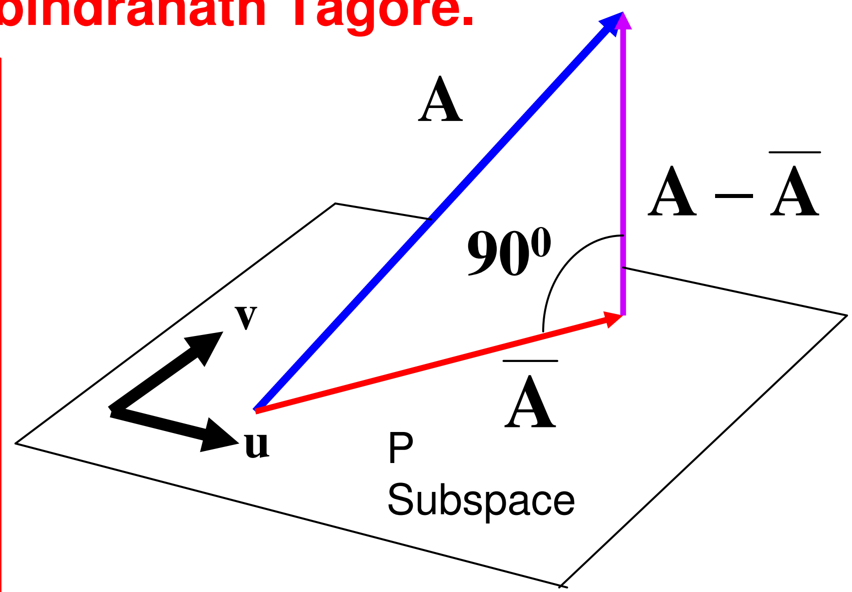
Optimal strain/stress recovery points (Prathap Points) result from the best-fit paradigm. It has been shown that the well known Gauss points for strain recovery are actually special cases of the Prathap points. For finer meshing, Prathap points progressively march towards Gauss points.

When Arts and Science met at the crossroads…



An extract from " *Sanchaita* " by Rabindranath Tagore.

আমারই চেতনার রঙে পান্না হল সবুজ,
চুনী উঠল রাঙা হয়ে।
আমি চোখ মেললুম আকাশে —
জ্বলে উঠল আলো
পূবে পশ্চিমে।
গোলাপের দিকে চেয়ে বললুম, সুন্দর —
সুন্দর হল সে।
তুমি বলবে, এ যে তত্ত্বকথা,
এ কবির বাণী নয়।
আমি বলব, এ সত্য,
তাই এ কাব্য।



– "সঞ্চয়িতা" – "আমি"
কবিগুরু রবীন্দ্রনাথ ঠাকুর
১৫ই জ্যৈষ্ঠ, ১৩৪৩
শান্তিনিকেতন